8 Appendix: An Implementation in *Julia*

In this appendix, we want to highlight a quick implementation of the iRB operator in the programming language *Julia*. This implementation has been used to generate all the images within the main text. [1]
We first import the *Plots* package for plotting and the *QuadGK* package for numerical integration.

```
using Plots
using QuadGK
```

We may then implement a function that computes the RB operator for given parameter functions.

```
function compute_irb_operator(hit_times,inv_l,q,s)
    function T(f)
        function Tf(x)
            (l,u) = hit_times(x)
            return first(quadgk(t -> q(t,inv_l(x)(t)), l, u)) +
                first(quadgk(t -> s(t,inv_l(x)(t)) * f(inv_l(x)(t)), l, u))
        end
        return Tf
    end
    return T
end
```

The parameter `hit_times` is a function taking $x \in X$ and returning the x-hit times $T_x = \{t \in [1,n] : x \in X_t\}$. For simplicity, we assume that T_x is an interval for every $x \in X$; represented as a pair `(l,b)`, where `l` is the lower bound of the interval and `b` is the upper one. The parameter `inv_l` returns for given $x \in X$ the function $t \mapsto {l_t}^{-1}(x)$. The function `q` requires $t \in [1,n]$ and $x \in X$ and returns a value in F. Similarly, `s` produces a real number for given $t \in [1,n]$ and $x \in X$.
The function `compute_irb_operator` returns the function `T` which takes an f and returns another function `Tf` that computes for given x the value $T(f)(x)$ using (3).
It is then straightforward to iterate this function `n` times, starting with the given function `start`.

```
function iterate_irb_operator(hit_times,inv_l,q,s,start,n)
    T = compute_irb_operator(hit_times,inv_l,q,s)
    fs = Any[start]
    for i in 1:n
        push!(fs,T(last(fs)))
    end
    return fs[2:end]
end
```

[1] A slight variation of the function has been used to plot the discontinuous examples in order to improve their visual appearance.

The return value of `iterate_irb_operator` is the list $[T(\texttt{start}), T(T(\texttt{start})), \dots]$.

Finally, we define a function to plot the resulting iterates. Its arguments are the same as before, except for the additional parameter `boundaries`, which is a pair representing the interval of x-values plotted in the graph.

```
function plot_approximations(hit_times,inv_l,q,s,start,n,boundaries)
    fs = iterate_irb_operator(hit_times,inv_l,q,s,start,n)
    p = plot(fs[1], xlims = boundaries, legend=false)
    for i in 2:length(fs)
        plot!(fs[i])
    end
    return p
end
```

To demonstrate the usage of these functions, we explain how to plot the iRB operator constructed in Example 4.4. We first have to compute the x-hit times for every $x \in X$. In general, if $h\colon [0,1] \to [0,1]$ is strictly monotonically increasing and continuous, then the x-hit time $T_x = \{t \in [1,2] : x \in X_t\}$ is an interval, because

$$\begin{aligned} t \in T_x &\iff l_t(0) \leq x \leq l_t(1) \\ &\iff \frac{1}{2}h(t - \lfloor t \rfloor) \leq x \leq \frac{1}{2}(1 + h(t - \lfloor t \rfloor)) \\ &\iff t \in \left[\inf_{t \in [1,2]} \{2x - 1 \leq h(t - \lfloor t \rfloor)\}, \sup_{t \in [1,2]} \{h(t - \lfloor t \rfloor) \leq 2x\} \right], \end{aligned}$$

where the last step uses the continuity of h. In the special case $h = \mathrm{id}_{[0,1]}$ considered in the example, we obtain

$$T_x = [2x, 2x+1] \cap [1,2] = \begin{cases} [1, 2x+1] & x \leq \frac{1}{2} \\ [2x, 2] & x > \frac{1}{2} \end{cases}.$$

Therefore, in *Julia* we define

```
function hit_times(x)
    if x <= 1/2
        return (1,2*x+1)
    else
        return (2*x,2)
    end
end
function inv_l(x)
    return t -> 2 * x - t + 1
end
```

and then we can generate Fig. 2 as follows:

```
plot_approximations(hit_times, inv_l, (t,x) -> 2*x*(t-1),
                    (t,x) -> 1/2*x*(t-1), x -> 0, 3, (0,1))
```

Similarly, replacing $\tilde{l}_2$ by the injective function

$$\tilde{l}_2 \colon [0,1] \to [0,1],\ x \mapsto 1 - \frac{1}{2}x^2$$

yields the extension (for $h = \mathrm{id}_{[0,1]}$)

$$l(t,x) = -\frac{1}{2}(t-1) \cdot x^2 + \frac{1}{2}(2-t) \cdot x + t - 1.$$

In this case, the quadratic polynomial l_t is not injective if and only if its maximum lies in the interval $(0,1)$, which turns out to be the case for $t \in (\frac{1}{3}, 1)$.
This shows that it need not be the case that l_t is injective for almost all t, even though all $\tilde{l}_i$ are injective.

In other words, for $i \in \{2,\dots,n-1\}$ and $t \in [i-\frac{1}{2}, i+\frac{1}{2})$, l_t is equal to $\tilde{l}_i$. Furthermore, l_t is equal to $\tilde{l}_1$ if $t \in [1, \frac{3}{2})$ and to $\tilde{l}_n$ for $t \in [n-\frac{1}{2}, n]$. Notice that the situation is slightly asymmetrical because l agrees with the $\tilde{l}_i$ for $i \in \{2,\dots,n-1\}$ on an interval of length 1 (after projecting onto the first component), whereas l is equal to $\tilde{l}_1$ or $\tilde{l}_n$ only on an interval of length $\frac{1}{2}$.

Applying the above construction (with respect to $h^{(k)}$) to $\tilde{l}_i$, $\tilde{q}_i$ and $\tilde{s}_i$ (and again replacing $\tilde{q}_1$ by $2 \cdot \tilde{q}_1$ and similarly for $\tilde{q}_n$, $\tilde{s}_1$ and $\tilde{s}_n$) yields a sequence of functions $l^{(k)}$, $q^{(k)}$ and $s^{(k)}$. However, two subtleties arise in this context. On the one hand, we need to assume that X is convex in order for $l^{(k)}$ to be a function $[1,n] \times X \to X$ (instead of having codomain E), which is the setting we are interested in (see Section 2). On the other hand, we need to ensure that the extensions $l^{(k)}$ have the property that for every $k \geq 2$, $l_t^{(k)}$ is injective for almost all $t \in [1,n]$. This is guaranteed if the set

$$\left\{ t \in [0,1] : (1-t) \cdot \tilde{l}_j + t \cdot \tilde{l}_{j+1} \text{ not injective} \right\}$$

is a null set for all $j \in \mathbb{N}_{n-1}$.
Hence, for every $k \geq 2$, we may construct a corresponding iRB operator $T^{(k)} \colon Z \to Z$. By Lemma 3.2, all $T^{(k)}$ are induced by continuous $l^{(k)}$, $q^{(k)}$ and $s^{(k)}$.

Hence, in order for $\mathbb{1}_{X_t}$ to be continuous at $x \in X$ for almost all $t \in [1,n]$, we must require that this set is a null set for every $x \in X$.
The second integrand $x \mapsto (s_t \circ {l_t}^{-1})(x) \cdot (f \circ {l_t}^{-1})(x) \mathbb{1}_{X_t}(x)$ is continuous at $x \in X$ for almost all $t \in [1,n]$ if additionally ${l_t}^{-1} \colon X_t \to X$ and $s_t \circ {l_t}^{-1} \colon X_t \to \mathbb{R}$ are continuous for almost all $t \in [1,n]$.