Complex Analysis 2

Lecture Notes

based on a lecture by Dr. P. Massopust

Marvin Jahn
mail@marvin-jahn.de

August 7, 2023

Contents

These are (unofficial) lecture notes for the lecture *Complex Analysis 2* held by Dr. P. Massopust at the Technical University Munich in the summer semester 2023.

All figures were drawn by Emna Marzouki, who also corrected countless mistakes.

1 Preliminaries

Lec 1 2023-04-19

We start by briefly recalling the most important definitions and theorems regarding complex analysis.
Without explicit mention, we will identify $\mathbb{C}$ and $\mathbb{R}^2$ as $\mathbb{R}$ vector spaces via the $\mathbb{R}$-isomorphism $\mathbb{C} \to \mathbb{R}^2,\ x + iy \mapsto (x, y)$.

Definition 1.1. A **region** is a nonempty open connected subset of $\mathbb{C}$.

Because an open subset of a locally path-connected space (like $\mathbb{C}$) is connected if and only if it is path-connected, we have the following statement.

Theorem 1.2. A nonempty open subset of $\mathbb{C}$ is a region if and only if it is path-connected.

Definition 1.3. Let $U \subset \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$ is called

(a) **holomorphic (on U)**, if it is complex differentiable at every point of U.

(b) **holomorphic (at $z_0 \in U$)**, if f is holomorphic on a neighborhood of z_0.

(c) **analytic on U**, if it can be expanded into a power series with positive radius of convergence at every point of U; i.e. for any $z_0 \in U$ there exists a neighborhood V of z_0 and coefficients $a_j \in \mathbb{C}$, such that $f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$ for all $z \in V$.

Definition 1.4. The **triangle (in $\mathbb{C}$) with corners** $z_0, z_1, z_2 \in \mathbb{C}$ is the convex hull of those three points; i.e. it is the compact set

$$\Delta = \{p_0 z_0 + p_1 z_1 + p_2 z_2 : p_0, p_1, p_2 \in [0, 1], p_0 + p_1 + p_2 = 1\}.$$

Theorem 1.5. Let $G \subset \mathbb{C}$ be a region and $z_0 \in G$. Every power series of the form $z \mapsto \sum_{j=0}^{\infty} a_j (z - z_0)^j$ (for fixed coefficients $a_j \in \mathbb{C}$) is a holomorphic function on the interior of its disk of convergence.

Theorem 1.6 ([Bor16, Thm 1.7.4])**.** If $f : U \to \mathbb{C}$ is holomorphic and $z_0 \in U$ with $f'(z_0) \neq 0$ then f is locally biholomorphic at z_0; that is, there exists an open neighborhood U_0 of z_0, such that the restriction $f : U_0 \to f(U_0)$ is holomorphic with holomorphic inverse.

As a consequence, on the open set $f'^{-1}(\mathbb{C}^\times)$ where its derivative does not vanish, any holomorphic function f is locally approximated by its first Taylor expansion

$$z \mapsto f(\xi) + f'(\xi) \cdot (z - \xi) \qquad \text{for } \xi \in f^{-1}(\mathbb{C}^\times),$$

which essentially just translates, rotates and scales a small shape. Therefore, for $\xi \in \mathbb{C}$ with $f'(\xi) \neq 0$, it locally preserve angles and orientations.

Theorem 1.7 (**Cauchy-Goursat-Morera-Weierstrass**)**.**
Let $G \subset \mathbb{C}$ be a region and $f : G \to \mathbb{C}$ a continuous function.
The following statements are equivalent:

(a) f is holomorphic on G.

(b) For any triangle $\Delta \subset G$, we have $\int_{\partial\Delta} f(z)dz = 0$.

(c) $f : G \to \mathbb{C}$ can be represented at every $z_0 \in G$ as a power series of the form $\sum_{j=0}^{\infty} a_j (z - z_0)^j$ with positive radius of convergence.

Furthermore, if f is holomorphic, the power series converges on the largest disk around z_0 contained within G and the coefficients can be computed as follows (for $r > 0$ small enough):

$$a_j = \frac{1}{2\pi i} \int_{\partial B_r(z_0)} \frac{f(\xi)}{(\xi - z_0)^{j+1}} d\xi.$$

Definition 1.8. Let $U \subset G$ be open. The set of all holomorphic functions $f\colon U \to \mathbb{C}$ is denoted by

$$\mathcal{O}(U) := \{f\colon U \to \mathbb{C} \text{ holomorphic}\}.$$

Proposition 1.9 (Principle of Analytic Continuation).
Let $G \subset \mathbb{C}$ be a region. For $f \in \mathcal{O}(G)$, if there is a nonempty open subset $U \subset G$, such that $f|_U = 0$, then $f = 0$. In particular, two functions agreeing on a nonempty open set are equal.

In words, the proposition states that holomorphic functions are uniquely determined by their local behavior on any open set.
In particular, the set of zeros of $f \in \mathcal{O}(G) \setminus \{0\}$ has an empty interior and such a f is never compactly supported.
Note that the statement is false if G is not connected, since the function could e.g. be constant on every connected component, attaining a diffferent value on each of them.

Definition 1.10. A sequence (x_n) in a topological space is called **eventually constant**, if there exists $N \in \mathbb{N}$, such that for all $n, m \geq N$, we have $x_n = x_m$.
For brevity, a sequence that is not eventually constant is called **non-trivial**.
Let $V \subset X$ be a subset of a topological space X. V **has an accumulation point (in X)**, if there exists a non-trivial sequence in V converging to some element of X.
V is called **discrete (in X)**, if it inherits the discrete topology from X; i.e. if for all $x \in V$, there exists a neighborhood U of x, such that $U \cap V = \{x\}$.

Unfortunately, in the literature the terminology and definitions vary. For example, [Bor16, Def 3.1.1] uses *discrete* for having no accumulation points and calls sets that inherit the discrete topology *locally finite*. On the other hand, only closed sets inheriting the discrete topology are called *locally finite* in [RS07]. It is also common to call the sets inheriting the discrete topology *isolated*.

The following lemma establishes the relation between these different notions.

Lemma 1.11. Let (X, d) be a metric space. A subset $V \subset X$ has no accumulation points if and only if it is discrete and closed.
In particular, a set has no accumulation point in itself if and only if it is discrete.

Proof. If V is not discrete, then there exists $x \in V$, such that for every $n \in \mathbb{N}_{>0}$, we have $B_{\frac{1}{n}}(x) \cap V \neq \{x\}$. Consequently, we can choose a non-trivial sequence converging to x, showing that V has an accumulation point.
Similarly, if V is not closed, there exists $x \notin V$ and a sequence $x_n \in V$ converging to x, so again V has an accumulation point.
On the other hand, if V is discrete and closed, the limit of a convergent sequence $(x_n) \in V$ must lie in V. Because V inherits the discrete topology, this shows that the sequence is eventually constant. □

It should be noted that these notions are quite subtle, they all depend on the ambient topological space X. For example, the set $\left\{\frac{1}{n} : n \in \mathbb{N}_{>0}\right\} \subset \mathbb{C}$ is discrete (in $\mathbb{C}$) but is not closed and clearly has an accumulation point.

Proposition 1.9 can be strengthened. Any two holomorphic functions defined on a region and agreeing on a set having an accumulation point are equal.

Theorem 1.12 (**Identity Theorem**).
Let $G \subset \mathbb{C}$ be a region, $f, g \in \mathcal{O}(G)$ and $S \subset G$ a subset having an accumulation point. If $f|_S = g|_S$, then $f = g$.

Proposition 1.13. For a region $G \subset \mathbb{C}$, the set $\mathcal{O}(G)$ with pointwise addition and multiplication is a commutative $\mathbb{C}$-algebra without zero divisors.

Proof. The commutative $\mathbb{C}$-algebra structure is easy to verify. To see that there are no zero divisors, suppose that $f \cdot g = 0$ and $f(z) \neq 0$ for some $z \in G$. By continuity, we can find an open neighborhood $U \subset G$, such that f is nonzero on all of U. Therefore, $g|_U = 0$ and Proposition 1.9 implies the claim. □

Theorem 1.14 (**Liouville's theorem**).
Every bounded holomorphic function $f \colon \mathbb{C} \to \mathbb{C}$ is constant.

Theorem 1.15 (**Open Mapping Theorem (of complex analysis)**).
Any non-constant holomorphic function $f \in \mathcal{O}(U)$ is an open map.
In particular, holomorphic functions preserve regions (the image of a region is another region).

Definition 1.16. Let $U \subset \mathbb{C}$ be open. A function $f \colon U \to \mathbb{C}$ is called **meromorphic (on** U**)**, if there is an open subset $V \subset U$, such that

(a) f is holomorphic on V.

(b) The complement $U \setminus V$ is discrete in U.

(c) For all points $p \in U \setminus V$, we have $\lim_{z\to p}|f(z)| = \infty$ (i.e. $\frac{1}{\lim_{z\to p}|f(z)|} = 0$).

The elements of $P(f) := U \setminus V$ are called the **poles** of f.
f is called **meromorphic at** $z_0 \in U$, if z_0 admits an open neighborhood on which f is meromorphic.

By Lemma 1.11, we may equivalently demand that $U \setminus V$ has no accumulation point in U.

If $f \colon U \to \mathbb{C}$ is a meromorphic function and is holomorphic at $p \in U$, then $\lim_{z\to p}|f(z)| = |f(p)|$, so that the set of poles of f is precisely the complement of the set of points where f is holomorphic. In particular, the set of poles $P(f)$ is uniquely determined.

Definition 1.17. Let $U \subset \mathbb{C}$ be open. The set of all meromorphic functions is denoted by

$$\mathcal{M}(U) := \{f \colon U \to \mathbb{C} \text{ meromorphic}\}.$$

Proposition 1.18. For a region $G \subset \mathbb{C}$, let $f \in \mathcal{O}(G) \setminus \{0\}$ and let Z denote the set of zeros of f. Then the function $\frac{1}{f} \colon G \to \mathbb{C}$ is meromorphic with set of poles Z.

Proof. Clearly, $g := \frac{1}{f}$ is holomorphic on $G \setminus Z$. Furthermore, Z is discrete and closed by Theorem 1.12 and Lemma 1.11. Thus for $p \in Z$, we can choose $r > 0$, such that $B_r(p) \subset G$ and $B_r(p) \cap Z = \{p\}$. Because g is holomorphic on $B_r(p) \setminus \{p\}$ and f is continuous, we have

$$\lim_{z \to p} |g(z)| = \lim_{z \to p} \frac{1}{|f(z)|} = \infty$$

and thus p is a pole of g. □

For a meromorphic function f with set of poles $P(f)$, the behavior of f on $P(f)$ is somewhat arbitrary. For example, the function $f\colon \mathbb{C}^\times \to \mathbb{C},\ z \mapsto \frac{1}{z}$ can be extended to the whole complex plane by setting its value at zero arbitrarily and it is meromorphic on $\mathbb{C}$ in any case. This freedom of choice may seem unsatisfactory; indeed it is resolved by adding an extra point ∞ and demanding that the set of poles are precisely those points mapping to ∞ (see Definition 2.3).
Alternatively, one can consider two meromorphic functions to be equal if they agree except possibly at their poles. For this, we identify two such functions (via an equivalence relation $\sim$) whenever they agree everywhere but on a discrete set.

Proposition 1.19. If $G \subset \mathbb{C}$ is a region, the set of equivalence classes $\mathcal{M}(G)/\sim$ with pointwise addition and multiplication is a field and $\mathbb{C}$-algebra with pointwise addition and multiplication.

Proof. We will prove a slightly more general result in Proposition 2.18. □

2 The Riemann Sphere

Lec 2
2023-04-26

In this section (which follows [JS87, Ch 1]), we define the *Riemann sphere* as a compact "extension" of the complex plane $\mathbb{C}$. We then transfer the key notions of holomorphy and meromorphy to this new setting.

2.1 Topological Properties of the Riemann Sphere

Let $S^2 := \{\xi \in \mathbb{R}^3 : \xi_1^2 + \xi_2^2 + \xi_3^2 = 1\}$ denote the 2-dimensional sphere in $\mathbb{R}^3$ and identify the z-plane $E := \{\xi \in \mathbb{R}^3 : \xi_3 = 0\}$ with $\mathbb{C}$.

One disadvantage of the complex plane is that it is not compact. This problem can be solved by constructing its *one-point compactification* (also called *Alexandroff compactification*).

We connect an arbitrary point $\xi \in S^2 \setminus \{N\}$ with the *north pole* $N := (0, 0, 1)$ of the sphere S^2 and denote its intersection with $\mathbb{C}$ by $\pi_0(\xi) \in \mathbb{R}^3$. This procedure defines the *stereographic projection* $\pi_0 \colon S^2 \setminus \{N\} \to \mathbb{C}$, which we now compute.

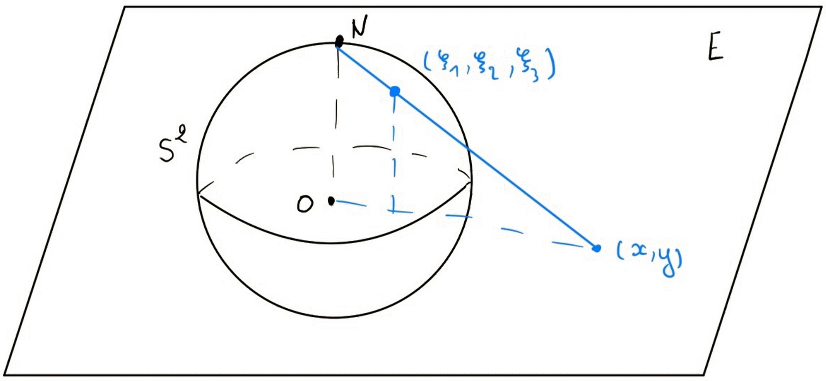

Figure 1: The *stereographic projection* π_0 mapping a point $(\xi_1, \xi_2, \xi_3) \in S^2 \setminus \{N\}$ to $(x, y) \in E$ (i.e. in $\mathbb{C}$).

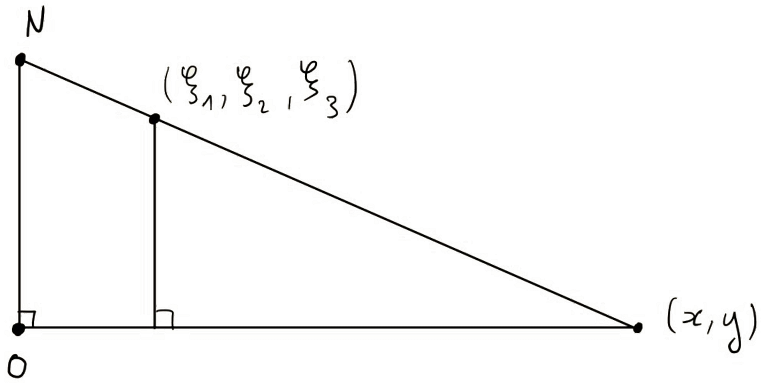

Figure 2: Projecting onto the x-plane or the y-plane allows us to compute the map π_0 concretely.

By projecting onto the x-plane or the y-plane, the similarity of the triangles yields

$$\frac{x}{1}=\frac{x-\xi_1}{\xi_3},\qquad \frac{y}{1}=\frac{y-\xi_2}{\xi_3},$$

which implies

$$x=\frac{\xi_1}{1-\xi_3},\qquad y=\frac{\xi_2}{1-\xi_3}.$$

Therefore, the stereographic projection is continuous and can be defined as follows.

Definition 2.1. Let $N=(0,0,1)\in S^2$ denote the north pole of the sphere S^2. The **stereographic projection** is

$$\pi_0\colon S^2\setminus\{N\}\to\mathbb{C},\ \xi\mapsto\frac{\xi_1}{1-\xi_3}+i\frac{\xi_2}{1-\xi_3}.$$

Since $\xi\in S^2$, we have

$$x^2+y^2+1=\frac{\xi_1^2+\xi_2^2+(1-\xi_3)^2}{(1-\xi_3)^2}=\frac{2-2\xi_3}{(1-\xi_3)^2}=\frac{2}{1-\xi_3}$$

and thus a straightforward calculation yields

$$\xi_1=\frac{2x}{1+x^2+y^2},\qquad \xi_2=\frac{2y}{1+x^2+y^2},\qquad \xi_3=\frac{x^2+y^2-1}{1+x^2+y^2}.$$

This gives the continuous inverse of the stereographic projection $\pi_0\colon S^2\setminus\{N\}\to\mathbb{C}$, establishing that it is a homeomorphism (where $S^2\setminus\{N\}\subset\mathbb{R}^3$ is equipped with the subspace topology).

Therefore, the stereographic projection shows that $S^2\setminus\{N\}$ is topologically the same as $\mathbb{C}$. However, if we consider the whole sphere S^2, then there is an extra point, namely the north pole N.

It is obvious from Fig. 1 (and can be confirmed by a calculation) that the stereographic projection π_0 maps the latitudinal circle $C(h):=\{\xi\in S^2:\xi_3=h\}$ for arbitrary $h\in[-1,1)$ to the planar circle $P(h):=\left\{z\in\mathbb{C}:|z|=\sqrt{\frac{1+h}{1-h}}\right\}$; that is, π_0 restricts to a homeomorphism $C(h)\to P(h)$. Therefore, the images of closer and closer circles around the north pole ($h\to1$) are circles of larger and larger radius in the complex plane. Therefore, the north pole N can be thought of as the unique "point at infinity". In particular, choosing a sequence $s_n\to N$ on the sphere corresponds to a sequence $(z_n)\in\mathbb{C}$ such that $|z_n|\to\infty$.

We summarize our results:

Lemma 2.2. The stereographic projection $\pi_0\colon S^2\setminus\{N\}\to\mathbb{C}$ is a homeomorphism and restricts to a homeomorphism $\{\xi\in S^2:\xi_3=h\}\to\left\{z\in\mathbb{C}:|z|=\sqrt{\frac{1+h}{1-h}}\right\}$ for any $h\in[-1,1)$.

Definition 2.3. Let $\hat{\mathbb{C}}:=\mathbb{C}\cup\{\infty\}$ denote the one-point compactification of $\mathbb{C}$. It is called the **Riemann sphere**.

This means that the open sets of $\hat{\mathbb{C}}$ are the open sets of $\mathbb{C}$ and the sets of the form $U\cup\{\infty\}$ for $U\subset\mathbb{C}$ open (in $\mathbb{C}$) with compact complement (in $\mathbb{C}$).

Lemma 2.4 (**Topological Properties of the Riemann sphere**)**.** The Riemann sphere $\hat{\mathbb{C}}$ is Hausdorff, compact and second countable.
Furthermore, the inclusion $\mathbb{C} \hookrightarrow \hat{\mathbb{C}}$ is an open embedding with dense image (i.e. $\mathbb{C} \subset \hat{\mathbb{C}}$ is an open dense subset and its subspace topology is precisely the usual topology on $\mathbb{C}$).

Proof. All of these properties are true more generally for the one-point compactification of any locally-compact, Hausdorff and second countable space (for the last part, see here). □

We extend π_0 to a bijection

$$\pi\colon S^2 \to \hat{\mathbb{C}},\ \ \pi|_{S^2\setminus\{N\}} = \pi_0,\ \ \pi(N) = \infty.$$

Lemma 2.5. Let $(s_n)_{n\in\mathbb{N}} \in S^2$ be a sequence and $(z_n)_{n\in\mathbb{N}} := \pi(s_n) \in \hat{\mathbb{C}}$ its image. Then

$$s_n \to N \iff |z_n| \to \infty \,\forall\, z_n \neq \infty \iff z_n \to \infty.$$

Proof. By Lemma 2.2, $s_n \to N$ is equivalent to $|z_n| \to \infty$ for those n with $z_n \neq \infty$. This in turn is equivalent to $z_n \notin \overline{B_k}(0)$ for all sufficiently large n and for all $k \in \mathbb{N}$. Since $\left\{\left(\mathbb{C} \setminus \overline{B_k}(0)\right) \cup \{\infty\} : k \in \mathbb{N}\right\}$ is a neighborhood basis of $\infty \in \hat{\mathbb{C}}$, the assertion follows. □

By this lemma, π is continuous at $N \in S^2$ and its inverse is continuous at $\infty \in \hat{\mathbb{C}}$. Because furthermore π restricts to a homeomorphism $S^2 \setminus \{N\} \to \mathbb{C}$ (by Lemma 2.2), the Riemann sphere really is topologically a 2-sphere. We summarize this result.

Theorem 2.6. The map $\pi\colon S^2 \to \hat{\mathbb{C}}$ defines a homeomorphism.

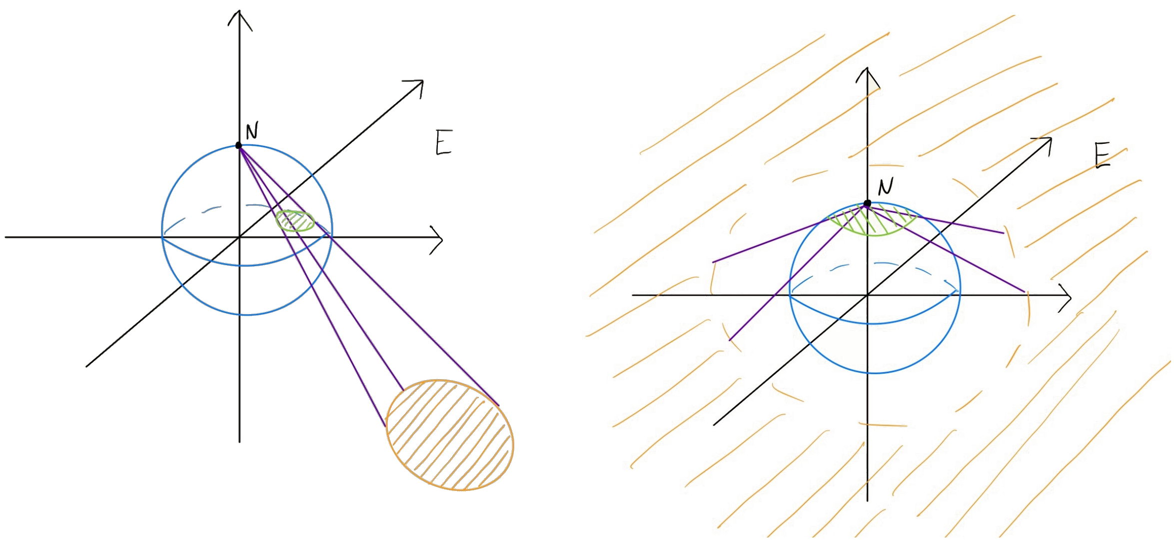

Figure 3: The left picture shows an open bounded set on the sphere (in green) and its image under π (the orange disk) lying in $\hat{\mathbb{C}}$. The right picture shows the image of an open set containing the north pole $N \in S^2$ (in green) and its image under π (in orange).

Equipping $S^2 \subset \mathbb{R}^3$ with the usual Euclidean metric, this homeomorphism induces the following metric on $\mathbb{C}$:

$$\chi(z, z') := d(\pi^{-1}(z), \pi^{-1}(z')).$$

The behavior of this metric can be computed explicitly with a lengthy calculation.

Theorem 2.7. The metric χ on $\hat{\mathbb{C}}$ defined above is given by

$$\chi\colon \hat{\mathbb{C}} \times \hat{\mathbb{C}} \to \mathbb{R},\ (z, z') \mapsto \begin{cases} \frac{2|z-z'|}{\sqrt{1+|z|^2}\sqrt{1+|z'|^2}} & z, z' \in \mathbb{C} \\ \frac{2}{\sqrt{1+|z|^2}} & z \in \mathbb{C}, z' = \infty \\ \frac{2}{\sqrt{1+|z'|^2}} & z = \infty, z' \in \mathbb{C} \\ 0 & z = z' = \infty. \end{cases}$$

It is called the **chordal metric**.

Because the topology on the sphere is that induced by the Euclidean metric and π is a homeomorphism, we obtain the following corollary.

Corollary 2.8. The topology on $\hat{\mathbb{C}}$ is metrizable for it is induced by the chordal metric. In particular, $(\hat{\mathbb{C}}, \chi)$ is a compact metric space and thus complete.
Equipping S^2 with the Euclidean metric and $\hat{\mathbb{C}}$ with the chordal metric makes $\pi\colon S^2 \to \hat{\mathbb{C}}$ into an isometric isomorphism.

2.2 Complex Analysis on the Riemann Sphere

Having introduced a metric on the Riemann sphere $\hat{\mathbb{C}}$, our next goal is to extend the notion of holomorphic and meromorphic functions from $\mathbb{C}$ to $\hat{\mathbb{C}}$.

Definition 2.9. The **inversion map** on $\hat{\mathbb{C}}$ is given by

$$J\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}},\ z \mapsto \begin{cases} 0 & z = \infty \\ \infty & z = 0 \\ \frac{1}{z} & \text{otherwise} \end{cases}.$$

Clearly, J is an involution (and thus bijective). A lengthy calculation shows that

$$\pi^{-1} \circ J \circ \pi\colon S^2 \to S^2$$

is linear and given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This means that if we identify S^2 with $\hat{\mathbb{C}}$ using the homeomorphism π, then the inversion map J is a rotation by π around the x-axis.

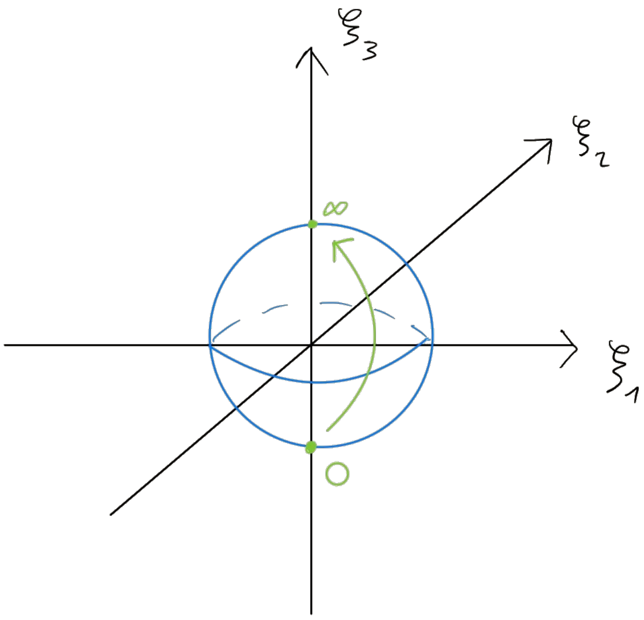

Figure 4: The action of the map J on the sphere S^2 is a rotation by π around the ξ_1-axis.

Corollary 2.10. The inversion map $J\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a homeomorphism.

This allows us to characterize the continuity of functions at $\infty \in \hat{\mathbb{C}}$.

Corollary 2.11. Let $U \subset \hat{\mathbb{C}}$ be an open neighborhood of $\infty \in \hat{\mathbb{C}}$ and $f\colon U \to \mathbb{C}$ a function. f is continuous at $\infty \in \hat{\mathbb{C}}$ if and only if $f \circ J$ is continuous at $0 \in \hat{\mathbb{C}}$ (i.e. $f(\infty) = \lim_{z\to 0}(f \circ J)(z)$).

This observation inspires us to translate the notions holomorphy and meromorphy from $\mathbb{C}$ to the Riemann sphere $\hat{\mathbb{C}}$.

Definition 2.12. Let $U \subset \hat{\mathbb{C}}$ be open and $f\colon U \to \mathbb{C}$ a function.
For $\infty \in U$, f is called **holomorphic (meromorphic) at** ∞ if $f \circ J\colon J(U) \to \mathbb{C}$ is holomorphic (meromorphic) at $0 \in J(U)$.
f is called **holomorphic** (**meromorphic**), if the restriction $f|_{U\cap\mathbb{C}}$ is holomorphic (meromorphic) and if furthermore f is holomorphic (meromorphic) at ∞ (assuming $\infty \in U$) .
Other terminology like **holomorphy at a point**, **pole**, **essential singularity**, **region** etc. is defined analogously.

In particular, a holomorphic function $f\colon U \to \mathbb{C}$ with $U \subset \hat{\mathbb{C}}$ is continuous, just like for functions defined on $\mathbb{C}$ instead of $\hat{\mathbb{C}}$.
Note that a holomorphic function $f\colon \hat{\mathbb{C}} \to \mathbb{C}$ is necessarily constant; for its image must be compact and thus $f|_{\mathbb{C}}$ must be constant by Theorem 1.14.

Because $\mathbb{C}$ is dense in $\hat{\mathbb{C}}$ (Lemma 2.4), any function $f\colon \mathbb{C} \to \mathbb{C}$ has at most one continuous extension $\hat{\mathbb{C}} \to \mathbb{C}$. By abuse of notation, it is common to define such a function only on $\mathbb{C}$. Then its value at $\infty \in \hat{\mathbb{C}}$ is chosen so that f becomes continuous; that is, by Corollary 2.11 and Lemma 2.5, we set

$$f(\infty) := \lim_{z\to 0}(f \circ J)(z) = \lim_{z\neq\infty, |z|\to\infty} f(z),$$

assuming this limit exists.

Example 2.13. (a) For the function

$$f\colon \hat{\mathbb{C}} \to \mathbb{C},\ z \mapsto \frac{1}{1+z^2},$$

the map $f \circ J(z) = \frac{z^2}{1+z^2}$ is holomorphic at $z = 0$ and has a zero of order 2 at $z = 0$. Therefore, f is holomorphic at $z = \infty$ and has a zero of order 2 at $z = \infty$.

(b) For the function ($c \in \mathbb{C}$ arbitrary)

$$f\colon \hat{\mathbb{C}} \to \mathbb{C},\ z \mapsto z^3, \infty \mapsto c$$

the map $f \circ J(z) = \frac{1}{z^3}$ is meromorphic at $z = 0$ with a pole of order 3. Thus, f is meromorphic with a pole of order 3 at $z = \infty$.

(c) For the sine ($c \in \mathbb{C}$ arbitrary)

$$f\colon \hat{\mathbb{C}} \to \mathbb{C},\ z \mapsto \sin(z), \infty \mapsto c$$

the composition $f \circ J(z) = \sin(\frac{1}{z})$ has an essential singularity at $z = 0$, so f has an essential singularity at $z = 0$ and thus is neither holomorphic nor meromorphic.

Lemma 2.14. Let $U \subset \hat{\mathbb{C}}$ be open and $f\colon U \to \mathbb{C}$ a function. f is holomorphic if and only if $f \circ J\colon J^{-1}(U) \to \mathbb{C}$ is holomorphic.

Proof. If $\infty \in U$, then by definition, f is holomorphic at ∞ if and only if the composition $f \circ J$ is holomorphic at $0 \in J^{-1}(U)$. Furthermore, if $0 \in U$, then since $f \circ J \circ J = f$, the composition $f \circ J$ is holomorphic at $\infty \in J^{-1}(U)$ if and only if f is holomorphic at 0. Finally, the restriction of $f \circ J$ to

$$J^{-1}(U) \cap (\mathbb{C} \setminus \{0, \infty\}) = \left\{z^{-1} : z \in U \setminus \{0, \infty\}\right\}$$

is given by $z \mapsto f(z^{-1})$ and thus the assertion follows since $\mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, z \mapsto \frac{1}{z}$ is biholomorphic. □

We can now extend Theorem 1.12 to the Riemann sphere $\hat{\mathbb{C}}$.

Theorem 2.15 (Identity Theorem for the Riemann Sphere).
Let $G \subset \hat{\mathbb{C}}$ be a region and $f\colon G \to \mathbb{C}$ a holomorphic function. Let $z_n \to z$ be a nonconstant convergent sequence in G with $z \in G$. If f vanishes on all of $\{z_n : n \in \mathbb{N}\}$, then $f = 0$.

Proof. We first assume that $z \neq \infty$. Then there exists $N \in \mathbb{N}$, such that for all $n \geq N$, $z_n \neq \infty$. Because $\hat{\mathbb{C}}$ is homeomorphic to the sphere S^2 (Theorem 2.6), $G' := G \setminus \{\infty\}$ is a region in $\mathbb{C}$. Furthermore, since $f \in \mathcal{O}(G')$ and $f(z_n) = 0$ for all $n \in \mathbb{N}$, $f|_{G'} = 0$ by Theorem 1.12. If $\infty \notin G$, this shows the assertion. Otherwise, the continuity of f yields $f = 0$.
We now consider the case $z = \infty$. Because z_n is not constant, we may replace z_n with a nonconstant subsequence and assume that $z_n \notin \{0, \infty\}$ for all $n \in \mathbb{N}$. By Lemma 2.14, $f \circ J\colon J^{-1}(U) \to \mathbb{C}$ is holomorphic and the homeomorphism J (Corollary 2.10) translates the nonconstant sequence $z_n \to z$ to a nonconstant sequence $J(z_n) \to J(z) = 0$. Since $(f \circ J \circ J)(z_n) = f(z_n) = 0$ for all $n \in \mathbb{N}$, Theorem 1.12 implies that $(f \circ J)|_{J^{-1}(U) \cap \mathbb{C}} = 0$, so $f|_{U \setminus \{0\}} = 0$ and the continuity of f establishes the claim. □

We now consider functions whose codomain is the Riemann sphere $\hat{\mathbb{C}}$ and extend the notion of holomorphy and meromorphy for function $\hat{\mathbb{C}} \to \mathbb{C}$ (see Definition 2.12) to the more general case $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. The Riemann sphere $\hat{\mathbb{C}}$ is the "correct" codomain to define meromorphy for functions. This is because for meromorphic mappings to ordinary $\mathbb{C}$, there is no natural value to map the poles to. Indeed in Definition 1.16, we allowed the value of the poles to be arbitrary. In contrast, in the setting of $\hat{\mathbb{C}}$, we can now demand that the preimage $f^{-1}(\infty)$ should precisely be the set of poles.

Definition 2.16. Let $U \subset \hat{\mathbb{C}}$ be open and $f\colon U \to \hat{\mathbb{C}}$ a function.
f is called **holomorphic (as a map to $\mathbb{C}$)**, if $\operatorname{im}(f) \subset \mathbb{C}$ and f is holomorphic as a map to $\mathbb{C}$.
f is called **meromorphic**, if it is meromorphic as a function into $\mathbb{C}$ with its set of poles equal to $f^{-1}(\infty)$. The notions of **holomorphy** and **meromorphy at a point** are defined analogously.
A point $z_0 \in U \setminus \{\infty\}$ is called a **pole of order** $k \in \mathbb{N}_{>0}$, if $J \circ f\colon V \to \mathbb{C}$ has a zero of order k at z_0, where V is a neighborhood of z_0 satisfying $V \subset f^{-1}(\hat{\mathbb{C}} \setminus \{0\}) \cap \mathbb{C}$.
If $\infty \in U$, it is called a **pole of order** $k \in \mathbb{N}_{>0}$, if $f \circ J\colon J(U) \to \hat{\mathbb{C}}$ has a pole of order k at $0 \in J(U)$.
The set of all meromorphic function on U is denoted by

$$\mathcal{M}(U) := \Big\{ f\colon U \to \hat{\mathbb{C}} \text{ meromorphic} \Big\}.$$

In particular, a point $z_0 \in U \setminus \{\infty\}$ is a pole of order k if and only if $z \mapsto (z - z_0)^k f(z)$ can be holomorphically extended to z_0 and is nonzero at z_0.
In fact, this definition can be generalized to Riemann surfaces (though note that these are generally assumed to be connected). $\mathbb{C}$ and $\hat{\mathbb{C}}$ provide fundamental examples of these complex manifolds, since the chart of $\mathbb{C}$ is just the identity $\operatorname{id}_{\mathbb{C}}$ and the charts of the Riemann sphere $\hat{\mathbb{C}}$ are $\operatorname{id}_{\mathbb{C}}$ and $J|_{\hat{\mathbb{C}} \setminus \{0\}}$. With this terminology, one can then define holomorphy for functions $S \to \mathbb{C}$ and meromorphy for functions $S \to \hat{\mathbb{C}}$, where S is an arbitrary Riemann surface (see here). There also exists a notion of holomorphy between two Riemann surfaces $f\colon S \to T$ by demanding that the composition $\psi \circ f \circ \phi^{-1}$ with charts ϕ and ψ is holomorphic. However, this definition is not equivalent to our definition of holomorphy (as a map to $\mathbb{C}$) above.

Clearly, any holomorphic (as a map to $\mathbb{C}$) function is meromorphic and any meromorphic function is continuous.

Example 2.17. (a) A constant function that is not infinity is holomorphic and the constant infinity function is not meromorphic.

(b) The identity $\operatorname{id}_{\hat{\mathbb{C}}}$ is meromorphic with a pole of order 1 at $\infty \in \hat{\mathbb{C}}$ and the inversion map $J\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic with a pole of order 1 at $0 \in \hat{\mathbb{C}}$.

Proposition 2.18. For an open set $U \subset \hat{\mathbb{C}}$ the set of meromorphic functions $\mathcal{M}(U)$ with pointwise addition and multiplication is a $\mathbb{C}$-algebra. If U is even a region, then it is a field.

Proof. To define $f + g$, $f \cdot g$ and $\lambda \cdot f$ for $f, g \in \mathcal{M}(G)$ and $\lambda \in \mathbb{C}$, one first defines the new function on the complement of the poles of f and g. This set is discrete since the union of two discrete sets is discrete. Then the function is extended to all of G by extending it

holomorphically. It is then straightforward to verify that this indeed defines a $\mathbb{C}$-algebra structure.
To see that it is a field if U is a region, it suffices to note that the multiplicative inverse of $f \in \mathcal{M}(G)\setminus\{0\}$ is $J \circ f \in \mathcal{M}(G)$ and it is seen to be meromorphic using Theorem 2.15. □

If U is not connected, $\mathcal{M}(U)$ is not necessarily a field. For example, consider the union of two balls $U = B_1(-2) \cup B_1(2)$ and set $f|_{B_1(-2)} := 0$, $f|_{B_1(2)} := 1$. The multiplicative inverse of this function would not be meromorphic, since it would be constant ∞ on $B_1(-2)$.

Lec 3 2023-05-03

We now study the solutions of equations on the Riemann sphere. More precisely, we consider an equation of the form $f(z_0) = \zeta$ for $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a nonconstant function and $\zeta \in \hat{\mathbb{C}}$.
We first consider a solution $z_0 \in \mathbb{C}$. If f is holomorphic (as a map to $\mathbb{C}$) at z_0 (so $\zeta \neq \infty$), then by Theorem 1.7 f is a local power series of the form

$$f(z) = \zeta + \sum_{j=k}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j,$$

where $k \in \mathbb{N}_{>0}$ is the smallest integer with $f^{(k)}(z_0) \neq 0$.
If instead f is meromorphic at z_0 with a pole of order $k \in \mathbb{N}_{>0}$ (so $\zeta = \infty$), then by definition $J \circ f$ has a zero of order k at z_0. In other words, we have $J \circ f(z) = (z - z_0)^k g(z)$ on a small enough punctured ball $B := B_r(z_0) \setminus \{z_0\}$ (with $r > 0$ chosen small enough such that f attains neither 0 nor ∞ on B), where g is a holomorphic function that does not vanish on $B_r(z_0)$. Therefore, we may write $f(z) = \frac{1}{(z-z_0)^k} \frac{1}{g(z)}$ with holomorphic $\frac{1}{g(z)}$ and thus f is locally equal to its Laurent series

$$f(z) = \sum_{j=-k}^{\infty} a_j (z - z_0)^j$$

for some $a_{-k} \neq 0$.

Definition 2.19. If $z_0 \in \hat{\mathbb{C}} \setminus \{\infty\}$ is a solution of $f(z_0) = \zeta$, then k as defined above is its **order** (or **multiplicity**).
If $z_0 = \infty \in \hat{\mathbb{C}}$ is a solution of $f(z_0) = \zeta$, then its **order** (or **multiplicity**) is the order of $f \circ J(z) = \zeta$ at $z = 0$.

By definition, the solutions of $f(z) = \infty$ of order k are precisely the poles of f of order k. In particular, $z_0 = \infty$ is a solution of $f(z_0) = \infty$ of order k if and only if $z = 0$ is a solution of $f \circ J(z) = \infty$ of order k. By the above, this means that we have a Laurent series $f \circ J(z) = \sum_{j=-k}^{\infty} b_j z^j$ for small enough $z \neq 0$, so plugging in $J(z)$ yields for large enough $z \neq \infty$:

$$f(z) = \sum_{j=-k}^{\infty} b_j \frac{1}{z^j} = \sum_{j=-\infty}^{k} b_{-j} z^j.$$

Definition 2.20. For $z_0 \in \hat{\mathbb{C}}$, the **principal part of** f **at** z_0 is the finite sum $\sum_{j=-k}^{-1} a_j (z - z_0)^j$ if $z_0 \neq \infty$ and it is $\sum_{j=1}^{k} b_{-j} z^j$ otherwise (with $a_j, b_j \in \mathbb{C}$ constructed as above).

Thus in a neighborhood of $z = 0$, we have a series representation

$$f \circ J(z) = \sum_{j=-k}^{\infty} a_j z^j$$

with $a_{-k} \neq 0$. By replacing z with $J(z)$, we obtain

$$f(z) = \sum_{j=-\infty}^{k} a_{-j} z^j$$

in a neighborhood of ∞. The **principal part** of f at ∞ is $\sum_{j=1}^{k} a_{-j} z^j$.

Definition 2.21. $z_0 \in \hat{\mathbb{C}}$ is called a **simple point of f at z_0** if it is a solution of order 1 of $f(z) = \zeta$ with $\zeta \coloneqq f(z_0)$. Otherwise, it is called a **multiple point**.

Example 2.22. As an example, we investigate the order of the zeros and poles of the function

$$f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}},\ z \mapsto \frac{z}{1+z^3}.$$

Clearly, 0 is a simple zero. Because $f \circ J(z) = \frac{z^2}{z^3+1}$ has a zero of order 2 at $z = 0$, the point $\infty \in \hat{\mathbb{C}}$ is a zero of order 2 at f. This is what we would expect, since f "looks like" $z \mapsto \frac{1}{z^2}$ for large $z \in \mathbb{C}$.
Let us now consider the poles of this function. $f(z) = \infty$ for $z \in \mathbb{C}$ means that $z^3 + 1 = 0$; i.e. $z \in \left\{-1, \exp\left(i\frac{\pi}{3}\right), \exp\left(-i\frac{\pi}{3}\right)\right\}$ are poles of order 1. ∞ is not a pole, because $J \circ f \circ J(z) = \frac{z^3+1}{z^2}$ does not vanish at $z = 0$.

This example generalizes in a straightforward way.

Lemma 2.23. Consider the rational function

$$f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}},\ z \mapsto \frac{p(z)}{q(z)},$$

where $p(z) = \sum_{j=0}^{n} a_j z^j$ and $q(z) = \sum_{j=0}^{m} b_j z^j \neq 0$ are two polynomials (of degree n and m, respectively) with coefficients $a_j, b_j \in \mathbb{C}$ and no common zeros.
Then f is meromorphic.
Furthermore, if f is not constant, its zeros and poles are characterized as follows:

(a) On $\mathbb{C}$, the zeros of f of order $k \in \mathbb{N}_{>0}$ are precisely the zeros of p of order k.

(b) On $\mathbb{C}$, the poles of f of order $k \in \mathbb{N}_{>0}$ are precisely the zeros of q of order k.

(c) $\infty \in \hat{\mathbb{C}}$ is a zero of f if and only if $m > n$, in which case its order is $m - n$.

(d) $\infty \in \hat{\mathbb{C}}$ is a pole of f if and only if $n > m$, in which case its order is $n - m$.

Proof. It is clear that f is meromorphic and (a) and (b) hold true. For (c), we note that $f \to 0$ for $z \to \infty$ means that $m > n$ and the order of this zero is $m - n$ because

$$f \circ J(z) = \frac{\sum_{j=0}^{n} a_j z^{-j}}{\sum_{j=0}^{m} b_j z^{-j}} = \frac{\sum_{j=0}^{n} a_j z^{m-j}}{\sum_{j=0}^{m} b_j z^{m-j}} = z^{m-n} \cdot \frac{\sum_{j=0}^{n} a_{n-j} z^j}{\sum_{j=0}^{m} b_{m-j} z^j}.$$

Since $\infty \in \hat{\mathbb{C}}$ is a pole of f of order k if and only if 0 is a zero of $J \circ f \circ J$ of order k and that is equivalent to $J \circ f = \frac{q}{p}$ having a zero at ∞ of order k, (d) follows from (c). $\square$

The rational functions form a field $\mathbb{C}(x)$, which extends $\mathbb{C}$, because the function mapping $c \in \mathbb{C}$ to the constant function with value c is a field homomorphism.

We now prove that the compactness of the Riemann sphere has the surprising consequence that nonconstant meromorphic functions $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ have finite preimages $f^{-1}(\zeta)$ for all $\zeta \in \hat{\mathbb{C}}$. This stands in direct contrast to the complex plane, which e.g. allows periodic functions like sin or exp.

Theorem 2.24. Let $f \in \mathcal{M}(\hat{\mathbb{C}})$ be nonconstant. For $\zeta \in \hat{\mathbb{C}}$, the equation $f(z) = \zeta$ has only finitely many solutions.

Proof. Fix $\zeta \in \hat{\mathbb{C}}$. We claim that for any $z_0 \in \hat{\mathbb{C}}$, there exists a neighborhood $U(z_0)$ of z_0, such that $f(z) \neq \zeta$ for all $z \in U(z_0) \setminus \{z_0\}$. Indeed, if $f(z_0) \neq \zeta$, such a neighborhood exists by continuity, so suppose that $f(z_0) = \zeta$. In case $\zeta = \infty$, the poles of f are the zeros of $J \circ f$ and if $\zeta \neq \infty$, the preimage $f^{-1}(\zeta)$ is precisely the set of zeros of $f - \zeta$. Therefore, the desired neighborhood exists by Theorem 2.15.
Because $\hat{\mathbb{C}}$ is compact, the cover $\hat{\mathbb{C}} = \bigcup_{z_0 \in \hat{\mathbb{C}}} U(z_0)$ must have a finite subcover (with say k elements), implying that the preimage $f^{-1}(\zeta)$ has at most k elements. □

Theorem 2.25. Let $f, g \in \mathcal{M}(\hat{\mathbb{C}})$ be two meromorphic functions. If they have poles at the same points and the same principal parts then they are equal up to constant in $\mathbb{C}$; that is, $f - g = c$ for some constant $c \in \mathbb{C}$.
This means that meromorphic functions on the Riemann sphere $\hat{\mathbb{C}}$ are determined up to additive constant by their principal parts.

Proof. The difference $h := f - g$ is continuous and since $\hat{\mathbb{C}}$ is compact, this implies that the image $\mathrm{im}(h)$ is compact. Since all principal parts cancel, $\mathrm{im}(h)$ is a compact subset of $\mathbb{C}$, so in particular bounded. Therefore Theorem 1.14 implies that h must be constant on $\mathbb{C}$ and by continuity, h is even constant on $\hat{\mathbb{C}}$. □

Because holomorphic functions $\hat{\mathbb{C}} \to \mathbb{C}$ (which we also called holomorphic functions $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as a map to $\mathbb{C}$) have no poles, we see that they must be constant.

Corollary 2.26. All holomorphic functions $f\colon \hat{\mathbb{C}} \to \mathbb{C}$ are constant.

The next result generalizes the fact that two complex polynomials in one variable having the same zeros with the same multiplicities are equal up to a multiplicative constant.

Theorem 2.27. Let $f, g \in \mathcal{M}(\hat{\mathbb{C}})$ and suppose that f and g have zeros and poles of the same orders at the same points of $\mathbb{C}$. Then $\frac{f}{g}$ is a constant in $\mathbb{C}$; i.e. $f = c \cdot g$ for some constant $c \in \mathbb{C}$.

Proof. If $f = 0$ or $g = 0$ the statement is clear, so we may assume that f and g are nonzero.
The function $h := \frac{f}{g}$ has a pole at ∞ if and only if $\frac{J \circ f \circ J}{J \circ g \circ J}$ vanishes at 0. Since f and g are meromorphic, we may write $J \circ f \circ J(z) = z^k \cdot \hat{f}(z)$ and $J \circ g \circ J(z) = z^l \cdot \hat{g}(z)$ for small $z \neq 0$, where $\hat{f}$ and $\hat{g}$ are holomorphic functions that are nonzero for small $z \neq 0$ and $k, l \in \mathbb{Z}$. By potentially interchanging f and g (i.e. considering the inverse $\frac{g}{f}$ instead of h), we may assume that $l \geq k$ and then

$$\frac{J \circ f \circ J(z)}{J \circ g \circ J(z)} = z^{k-l} \cdot \frac{\hat{f}}{\hat{g}}$$

does not vanish at 0, showing that $h(\infty) \in \mathbb{C}$. We now show that $h(\mathbb{C}) \subset \mathbb{C}$, which with the above implies that $\operatorname{im}(h) \subset \mathbb{C}$.
If $z_0 \in \mathbb{C}$ is a zero of f of order k, then f is of the form $f(z) = (z - z_0)^k \hat{f}(z)$ on a small punctured disk $B_r(z_0) \setminus \{z_0\}$ (with $r > 0$), where $\hat{f}$ is a holomorphic function that is nonzero on all of $B_r(z_0)$. By assumption, g can similarly be written as $g(z) = (z-z_0)^k \hat{g}(z)$, so their fraction $h(z) = \frac{f(z)}{g(z)}$ is holomorphic at z_0. An analogous argument shows that h is also holomorphic at all poles of f (or g).
We conclude (as in the proof before), that by compactness of the Riemann sphere $\operatorname{im}(h) \subset \mathbb{C}$ must be bounded, which implies that h is constant by Theorem 1.14. □

Note that in case $f = 0$, we have $g = 0$ and the fraction $\frac{0}{0}$ is by convention understood to be the constant 1 function.

We have observed in Lemma 2.23 that rational functions on the Riemann sphere $\hat{\mathbb{C}}$ are meromorphic. In fact, the opposite is true, too.

Theorem 2.28. The meromorphic functions $\mathcal{M}(\hat{\mathbb{C}})$ are precisely the rational functions.

Proof. As mentioned, any rational function is meromorphic by Lemma 2.23.
On the other hand, let $f \in \mathcal{M}(\hat{\mathbb{C}}) \setminus \{0\}$ be a nonzero meromorphic function. Then its set of zeros N and its set of poles P are both finite by Theorem 2.24. Let $M(x) > 0$ denote the order of $x \in N \cup P$. Then f and the rational function

$$r(z) := \frac{\prod_{n \in N} (z - n)^{m(n)}}{\prod_{p \in P} (z - p)^{m(p)}}$$

share the same zeros and poles of the same orders at the points of $\mathbb{C}$, so $f = c \cdot r$ for some constant $c \in \mathbb{C}$ by Theorem 2.27; that is, f is a rational function. □

3 Elementary Transcendental Functions

3.1 Holomorphic Extensions

We briefly discuss when and how one can extend a function defined on an interval to a holomorphic function on a region in $\mathbb{C}$.

Definition 3.1. Let $(a,b) \subset \mathbb{R}$ be an open interval, $G \subset \mathbb{C}$ a region with $(a,b) \subset G$ and $f\colon (a,b) \to \mathbb{R}$ a function. If $F\colon G \to \mathbb{C}$ is holomorphic with $F|_{(a,b)} = f$, then F is called the **holomorphic extension** of f.

Note that by Theorem 1.12, the holomorphic extension of f on a given region G is, if existent, unique. There are also other notions of *holomorphic extension* in usage.

Theorem 3.2. A function $f\colon (a,b) \to \mathbb{R}$ has a holomorphic extension if and only if f is real analytic; i.e. if for all $x_0 \in (a,b)$, there exists $r(x_0) > 0$ and coefficients $a_j \in \mathbb{R}$ such that $f(x) = \sum_{j=0}^{\infty} a_j(x-x_0)^j$ for all $x \in (x_0 - r, x_0 + r)$.

Proof. If there exists a holomorphic extension F of f, then F has a power series expansion (in $\mathbb{C}$) and the coefficients $a_j = \frac{1}{j!}F^{(j)}(z)\Big|_{z=x_0}$ lie in $\mathbb{R}$.
On the other hand, suppose that f is analytic (on $\mathbb{R}$) and let $x_0 \in (a,b)$. By definition, there exists $r(x_0) > 0$ and coefficients $a_j(x_0) \in \mathbb{R}$ (both dependent on x_0), such that $f(x) = \sum_{j=0}^{\infty} a_j(x_0)(x-x_0)^j$ for all $x \in (x_0 - r, x_0 + r)$. Because this power series converges on all of $B_{r(x_0)}(x_0)$, the holomorphic function

$$F_{x_0}\colon B_{r(x_0)}(x_0) \to \mathbb{C},\ z \mapsto \sum_{j=0}^{\infty} a_j(x_0)(z-x_0)^j$$

agrees with f on $I(x_0) := (x_0 - r(x_0), x_0 + r(x_0))$.
It remains to show that for $x_0, x_1 \in (a,b)$ with $x_0 < x_1$, the coefficients $a_j(x_0)$ and $a_j(x_1)$ agree for all $j \in \mathbb{N}$, since then the collection $\{F_{x_0} : x_0 \in (a,b)\}$ gives rise to a holomorphic extension F of f on the region $\bigcup_{x\in(a,b)} B_{r(x)}(x)$.
Indeed, because the interval $[x_0, x_1] \subset \mathbb{R}$ is compact, the cover $\bigcup_{x\in[x_0,x_1]} I(x)$ admits a finite subcover $\bigcup_{u=1}^{k} I(y_u)$ for some $y_u \in (a,b)$ with $I(y_u) \not\subset I(y_v)$ for $u \neq v$ and $y_1 < y_2 < \cdots < y_k$. For $u \in \{1,\dots,k-1\}$, it follows that $I(y_u) \cap I(y_{u+1})$ is a nonempty interval on which F_{y_u} and $F_{y_{u+1}}$ agree. By Theorem 1.12, they even agree on the open set $B_{r(y_u)}(y_u) \cap B_{r(y_{u+1})}(y_{u+1}) \subset \mathbb{C}$, implying that $a_j(y_u) = a_j(y_{u+1})$ for all $j \in \mathbb{N}$. Since this applies to every $u \in \{1,\dots,k-1\}$, we have $a_j(y_1) = a_j(y_k)$ and by again applying Theorem 1.12, we see that $a_j(y_1) = a_j(x_0)$ and $a_j(y_k) = a_j(x_1)$, so $a_j(x_0) = a_j(x_1)$ for all $j \in \mathbb{N}$, as desired. □

Lec 4 2023-05-10

Example 3.3. The exponential function

$$\exp\colon \mathbb{C} \to \mathbb{C},\ z \mapsto \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

is the holomorphic extension of the real analytic function $\exp\colon \mathbb{R} \to \mathbb{R}$. Here we used that the radius of convergence of this power series is infinite. Similarly,

$$\cos(z) := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}, \quad \sin(z) := \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}$$

are the extensions of the real analytic functions cosine and sine, respectively. The hyperbolic cosine and hyperbolic sine are given by

$$\cosh(z) = \frac{1}{2}(\exp(z) + \exp(-z)), \quad \sinh(z) = \frac{1}{2}(\exp(z) - \exp(-z)).$$

and satisfy

$$\cosh(z) = \cos(iz), \quad \sinh(z) = -i\sin(iz).$$

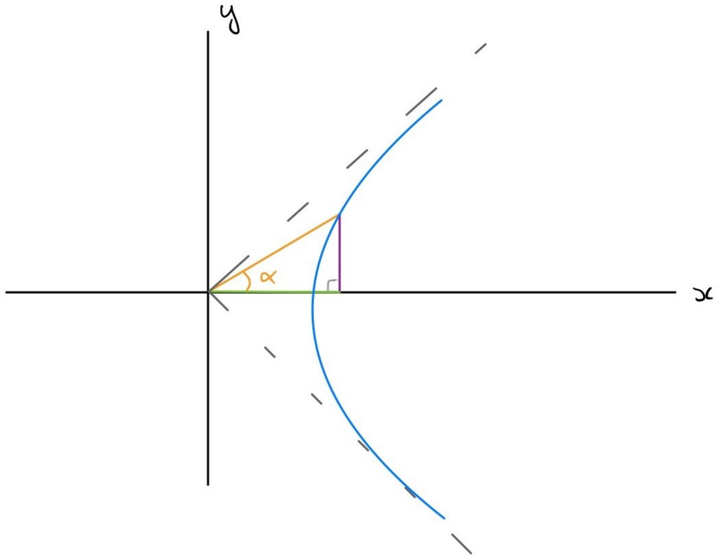

Figure 5: The x-coordinate of a fixed point on the parabola $x^2 - y^2 = 1$ (in blue) is $\cosh(\alpha)$ (in green). Similarly, its y-coordinate is $\sinh(\alpha)$ (in purple). This is analogous to the characterization of sine and cosine on the unit circle and explains why the functions are called hyperbolic.

The identity theorem (Theorem 1.12) gives rise to *principle of permanence*, which allows us to transfer many identities that we know are true on $\mathbb{R}$ to a corresponding identity for the holomorphic extension on $\mathbb{C}$. It is demonstrated by the following proposition.

Proposition 3.4. The exponential function $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is a group homomorphism; i.e. $\exp(z + z') = \exp(z) \cdot \exp(z')$ for all $z, z' \in \mathbb{C}$.

Proof. For fixed $z' \in \mathbb{R}$, the functions $\mathbb{C} \to \mathbb{C}$

$$z \mapsto \exp(z + z'), \quad z \mapsto \exp(z) \cdot \exp(z')$$

are holomorphic and agree on $\mathbb{R} \subset \mathbb{C}$, so by Theorem 1.12, they agree on all of $\mathbb{C}$. Fixing $z \in \mathbb{R}$ instead and applying the same argument yields the claim. □

For $z = x + iy \in \mathbb{C}$ (with $x, y \in \mathbb{R}$), we have

$$\exp(z) = \exp(x) \cdot \exp(iy) = \exp(x) \cdot (\cos(y) + i\sin(y)). \tag{1}$$

3.2 Periodic Functions

To express the periodicity of the exponential function, we characterize periodicity of functions in much generality.

Definition 3.5. Let G be an abelian group and T an arbitrary set. A function $f\colon G \to T$ is called **periodic with period** $a \in G \setminus \{0\}$, if it is invariant under the action of a; i.e. if we have $f(z) = f(z + a)$ for all $z \in G$.
The set of periods (together with $0 \in G$)

$$P := \{a \in G : f \text{ periodic with period } a\} \cup \{0\} \subset G$$

is a group. If it is cyclic with generator $a \in G$, then f **simply periodic** with **primitive period** a.

Example 3.6.

(a) The group of periods of the *Dirichlet function* $\mathbb{1}_{\mathbb{Q}}\colon \mathbb{R} \to \mathbb{R}$ is $\mathbb{Q}$, so it is not simply periodic.

(b) Sine and cosine (as functions $\mathbb{C} \to \mathbb{C}$) are simply periodic with primitive period 2π (or -2π), because their restrictions to the real line have this property and the periodicity extends by Theorem 1.12 as the holomorphic functions $z \mapsto \sin(z)$ and $z \mapsto \sin(z + 2\pi)$ agree on $\mathbb{R}$.

(c) The exponential function $\exp\colon \mathbb{C} \to \mathbb{C}$ is periodic with primitive period $2\pi i$ (or $-2\pi i$), because

$$\exp(z) = \exp(z + a) = \exp(z) \cdot \exp(a) \quad \forall\ z \in \mathbb{C}$$

is equivalent to $\exp(a) = 1$; i.e. to $a \in 2\pi i\mathbb{Z}$.

By definition, a periodic function $f\colon G \to T$ with group of periods P descends to a function $G/P \to T$; i.e. it factors uniquely through the canonical projection $\pi\colon G \to G/P$:

$$\begin{array}{ccc} G & \xrightarrow{\ f\ } & T \\ \big\downarrow{\scriptstyle\pi} & \nearrow & \\ G/P & & \end{array}$$

Moreover, if all groups are topological groups and f is continuous, the induced map is also continuous.

This means that a continuous periodic function $f\colon G \to T$ can equally well be viewed as a function $G/P \to T$. For example, $\sin\colon \mathbb{R} \to \mathbb{R}$ and $\cos\colon \mathbb{R} \to \mathbb{R}$ are continuous on the topological group $\mathbb{R}$ (with ordinary addition) and thus descend to functions $\mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ and it can be shown that $\mathbb{R}/2\pi\mathbb{Z}$ is homeomorphic to the circle S^1. In this sense, sine and cosine are really defined "on the circle", which is why the trigonometric functions are also called *circular functions*. This point of view is sometimes very convenient, for instance in Fourier analysis.
Similarly, the exponential function can be thought of as being defined on the (infinite) cylinder $\mathbb{R} \times S^1$.

Definition 3.7. The quotient group G/P is called the **period quotient group of** G.

3.3 The Complex Logarithm

Geometrically, the argument $\arg(z)$ of a complex number $z \in \mathbb{Z}$ is the angle between the real axis and the point z (when connected to the origin). In particular, it is clear that it is only unique in $\mathbb{R}/2\pi\mathbb{Z}$.

Definition 3.8. The set-valued **argument function** is

$$\arg\colon \mathbb{C}^\times \to \mathcal{P}(\mathbb{R}),\ z \mapsto \left\{ r \in \mathbb{R} : \frac{z}{|z|} = \exp(ir) \right\} = \operatorname{Arg}(z) + 2\pi\mathbb{Z},$$

where $\operatorname{Arg} : \mathbb{C}^\times \to \mathbb{R}$ is the function choosing the unique representative in $(-\pi, \pi]$. Arg is called the **principal value of** arg.

Note that Arg (and arg as well) is really a function from the unit sphere S^1, since it factors as

$$\mathbb{C}^\times \xrightarrow{z \mapsto \frac{z}{|z|}} S^1 \xrightarrow{\operatorname{Arg}} \mathbb{R}.$$

Proposition 3.9. The principal value of arg is given by

$$\operatorname{Arg} : \mathbb{C}^\times \to \mathbb{R},\ x + iy \mapsto \begin{cases} \arctan\left(\frac{y}{x}\right) & x > 0 \\ \arctan\left(\frac{y}{x}\right) + \pi & x < 0,\ y \geq 0 \\ \arctan\left(\frac{y}{x}\right) - \pi & x < 0,\ y < 0 \\ \frac{\pi}{2} & x = 0,\ y > 0 \\ -\frac{\pi}{2} & x = 0,\ y < 0 \end{cases}.$$

It is not continuous at any point on the negative real axis $(-\infty, 0) \subset \mathbb{C}^\times$, but it is totally differentiable on

$$\mathbb{C}^- := \mathbb{C}^\times \setminus (-\infty, 0) = \mathbb{C} \setminus (-\infty, 0]$$

with

$$\partial_x \operatorname{Arg}(x, y) = -\frac{y}{x^2 + y^2}, \qquad \partial_y \operatorname{Arg}(x, y) = \frac{x}{x^2 + y^2}.$$

Proof. The formula for Arg follow from elementary geometry on the unit circle. Furthermore, it is clear that Arg is not continuous at $x \in \mathbb{R} \subset \mathbb{C}$ with $x < 0$ as

$$\lim_{y \to 0, y > 0} \operatorname{Arg}(x + iy) = \lim_{y \to 0, y > 0} \arctan\left(\frac{y}{x}\right) + \pi = \pi \neq -\pi = \lim_{y \to 0, y < 0} \operatorname{Arg}(x + iy).$$

The claimed derivatives are easily confirmed on the open subsets $\{x + iy : x > 0, y \in \mathbb{R}\}$, $\{x + iy : x < 0, y > 0\}$ and $\{x + iy : x < 0, y < 0\}$ using that $\partial \arctan(t) = \frac{1}{1+t^2}$. Finally, on $\{iy : y > 0\}$ and $\{iy : y < 0\}$, using the definition of the derivative and L'Hôpital's rule yields the claim. □

By (1), we have for $z = x + iy \in \mathbb{C}$ $(x, y \in \mathbb{R})$:

$$|\exp(z)| = \exp(x), \qquad \arg(\exp(z)) = y + 2\pi\mathbb{Z}, \qquad \overline{\exp(z)} = \exp(\bar{z}).$$

Because the exponential function $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is periodic and thus in particular not injective, there exists no global left-inverse. However, since it is surjective, a right-inverse (not necessarily holomorphic) must exist (just like for sine and cosine). In order to compute such a right-inverse l, we use polar coordinates to write $z = |z| \exp(i \operatorname{Arg}(z)) \in \mathbb{C}^\times$, and note that by uniqueness of this representation, the equation

$$z = \exp(l(z)) = \exp(\Re(l(z))) \cdot \exp(i\Im(l(z)))$$

is equivalent to

$$l(z) = \log |z| + i \cdot (\operatorname{Arg}(z) + 2\pi k(z)),$$

where $k\colon \mathbb{C}^\times \to \mathbb{Z}$ can be chosen arbitrarily. In order for such a right-inverse to be continuous on $\mathbb{C}^-$, k must be continuous and thus constant, because the subspace topology $\mathbb{Z} \subset \mathbb{C}$ is the discrete one. Therefore, the continuous right-inverses of $\exp\colon \mathbb{C} \to \mathbb{C}^-$ are precisely

$$l\colon \mathbb{C}^- \to \mathbb{C},\ z \mapsto \log(|z|) + i \cdot (\operatorname{Arg}(z) + 2\pi k)$$

for arbitrary $k \in \mathbb{Z}$.
Furthermore, since Arg is not continuous on all of $\mathbb{C}^\times$ but the other functions occuring in l are, there exists no continuous right-inverse of $\exp\colon \mathbb{C} \to \mathbb{C}^\times$.

Definition 3.10. For $k \in \mathbb{Z}$, the function

$$\log_k\colon \mathbb{C}^\times \to \mathbb{C},\ z \mapsto \log(|z|) + i \operatorname{Arg}(z) + 2\pi i k$$

is called the **k-th branch of the logarithm** and is often abbreviated by log. The case $k = 0$

$$\operatorname{Log}\colon \mathbb{C}^\times \to \mathbb{C},\ z \mapsto \log(|z|) + i \operatorname{Arg}(z)$$

is called the **principal** (or **main**) **branch of the logarithm**.

By verifying the Cauchy-Riemann equations (using Proposition 3.9) all branches of log are seen to be holomorphic on $\mathbb{C}^-$. By the above, they cannot holomorphically be extended to all of $\mathbb{C}^\times$.

Proposition 3.11. Every branch of log is holomorphic on $\mathbb{C}^-$.

For $k \in \mathbb{Z}$, the *periodic strips*

$$S_k := \{x + iy : x \in \mathbb{R}, y \in (-\pi + 2\pi k, \pi + 2\pi k]\}$$

cover $\mathbb{C}$ and each of them provides a choice of representatives of the period quotient group $\mathbb{C}/(2\pi i\mathbb{Z})$ of exp. In particular, all strips S_k have the same image, namely $\mathbb{C}^\times$.

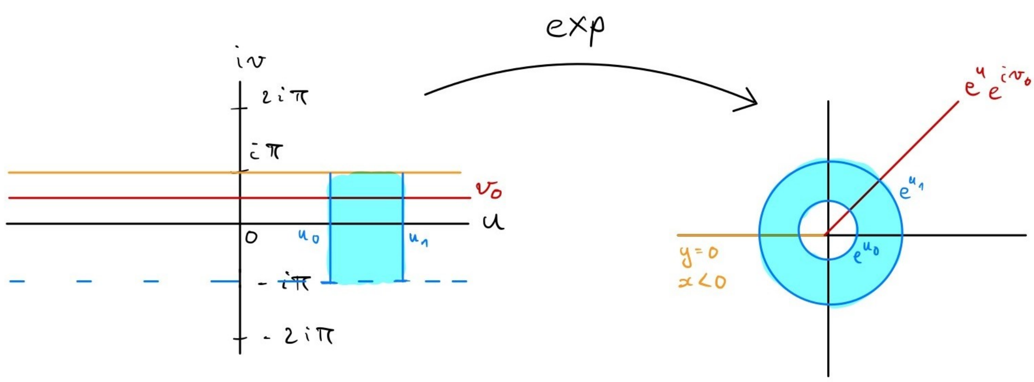

Figure 6: The strip S_0 and the image of various shapes under the exponential exp.

The "dual" rule to Proposition 3.4, does not generally hold true for the logarithm in the complex plane. This subtlety essentially stems from the discontinuity of log at $(-\infty, 0)$ (which in turn arises from the discontinuity of Arg).

Lemma 3.12. For $a, b \in \mathbb{C}^\times$, we have

$$\operatorname{Log}(a \cdot b) = \begin{cases} \operatorname{Log}(a) + \operatorname{Log}(b) & \operatorname{Arg}(a) + \operatorname{Arg}(b) \in (-\pi, \pi] \\ \operatorname{Log}(a) + \operatorname{Log}(b) - 2\pi i & \operatorname{Arg}(a) + \operatorname{Arg}(b) > \pi \\ \operatorname{Log}(a) + \operatorname{Log}(b) + 2\pi i & \operatorname{Arg}(a) + \operatorname{Arg}(b) \le -\pi. \end{cases}$$

Proof. Writing out the definition with a, b in polar coordinates and using that the rule is true on $\mathbb{R}_+$, we immediately see that

$$\operatorname{Log}(a \cdot b) = \operatorname{Log}(a) + \operatorname{Log}(b) \iff \operatorname{Arg}(a \cdot b) = \operatorname{Arg}(a) + \operatorname{Arg}(b). \qquad \square$$

We noted above that Arg (and thus all branches of log) are discontinuous at all points in $(-\infty, 0) \subset \mathbb{C}$ and that no global continuous right-inverse can exist. Indeed, the k-th branch of the log satisfies

$$\lim_{z \to z_0, \Im(z) > 0} \log_k(z) = \log(|z|) + (\pi + 2\pi k)i, \qquad \lim_{z \to z_0, \Im(z) < 0} \log_k(z) = \log(|z|) + (-\pi + 2\pi k)i$$

for all $z_0 \in (-\infty, 0)$. For arbitrary $k \in \mathbb{Z}$, these limits agree locally if we use the k-th branch when $\Im(z) \ge 0$ and the $(k+1)$-st branch otherwise. This idea leads to the Riemann surface of log, see wikipedia. Lec 5 2023-05-17

3.4 Complex Roots

For $n \in \mathbb{N}$, $n > 1$, we consider the n-th monomial $m\colon \mathbb{C}^\times \to \mathbb{C}^\times, z \mapsto z^n$. The preimage $m^{-1}(1)$ (i.e. the roots of $z \mapsto z^n - 1$) consists precisely of the n roots of unity (and form a cyclic group)

$$m^{-1}(1) = \left\{ \exp\left(2\pi \frac{k}{n} i\right) : k \in \{0, \dots, n-1\} \right\}.$$

The preimage of $c \in S^1$ with polar coordinates $c = \exp(i \operatorname{Arg}(c))$ is given by

$$m^{-1}(c) = m^{-1}(1) \cdot \exp\left(\frac{\operatorname{Arg}(c)}{n} i\right) = \left\{ \exp\left(\frac{2\pi k + \operatorname{Arg}(c)}{n} i\right) : k \in \{0, \dots, n-1\} \right\}.$$

More generally, by writing $c \in \mathbb{C}^\times$ in polar coordinates $c = |c| \exp(i \operatorname{Arg}(c))$, it follows that

$$\begin{aligned} m^{-1}(c) = \sqrt[n]{|c|} \cdot m^{-1}\left(\frac{c}{|c|}\right) &= \left\{ \sqrt[n]{|c|} \cdot \exp\left(\frac{2\pi k + \operatorname{Arg}(c)}{n} i\right) : k \in \{0, \dots, n-1\} \right\} \\ &= \left\{ \exp\left(\frac{1}{n} \log_k(c)\right) : k \in \{0, \dots, n-1\} \right\}. \end{aligned}$$

Therefore, (just like $\exp\colon \mathbb{C} \to \mathbb{C}^\times$) m admits no left-inverse, and any right-inverse $l\colon \mathbb{C}^\times \to \mathbb{C}^\times$ is of the form $l(z) = \exp\left(\frac{1}{n} \log_{k(z)}(z)\right)$ for some function $k\colon \mathbb{C}^\times \to \mathbb{Z}$. Restricting l to $\mathbb{C}^-$, all functions occuring in its definition are continuous and thus $l\colon \mathbb{C}^- \to \mathbb{C}^\times$ is continuous if and only if

$$\mathbb{C}^- \to \mathbb{C}^\times,\ z \mapsto \sqrt[n]{|z|} \cdot \exp\left(2\pi \frac{k(z)}{n} i\right) \cdot \exp\left(\frac{\operatorname{Arg}(z)}{n} i\right)$$

is continuous; i.e. if and only if k is constant (because again the subspace topology $\mathbb{Z} \subset \mathbb{C}$ is the discrete one).

Definition 3.13. For $n \in \mathbb{N}_{>0}$ and $k \in \mathbb{Z}$, the function

$$\sqrt[n]{\cdot}_k \colon \mathbb{C}^\times \to \mathbb{C}^\times, z \mapsto \exp\left(\frac{1}{n}\log_k(z)\right) = \sqrt[n]{|z|} \cdot \exp\left(\frac{2\pi k + \operatorname{Arg}(z)}{n} i\right)$$

is called the **k-th branch of the n-th root** and is often abbreviated by $\sqrt[n]{\cdot}$.

Note that $\sqrt[n]{\cdot}_k = \sqrt[n]{\cdot}_l$ if and only if $k \equiv l \mod n$. In particular, there are precisely n (distinct) branches of the n-th root. Because $\log_k$ is holomorphic on $\mathbb{C}^-$ (Proposition 3.11), the same holds true for $\sqrt[n]{\cdot}_k$.

Proposition 3.14. Every branch of the n-th root is holomorphic on $\mathbb{C}^-$.

As before, we observe that for $z_0 \in (-\infty, 0)$:

$$\lim_{z \to z_0, \Im(z) > 0} \sqrt[n]{z}_k = \lim_{z \to z_0, \Im(z) < 0} \sqrt[n]{z}_{(k+1 \mod n)},$$

Using this observation, one can construct the Riemann surface of $\sqrt[n]{\cdot}$, see wikipedia.

4 Homology in the Complex Plane

In this section, we highlight homology in the complex plane and state a version of Cauchy's Integral theorem for differentiable 1-forms. This requires quite a lot of terminology. Many of the definitions we give can be generalized to arbitrary manifolds.

Throughout this section, $U \subset \mathbb{C}$ denotes an open subset and for $k \in \mathbb{N} \cup \{\infty\}$, we write $C^k(U, \mathbb{C})$ for the $\mathbb{C}$-algebra of k-times continuously differentiable functions $U \to \mathbb{C}$.

4.1 The Cotangent Space

Definition 4.1. Let $U \subset \mathbb{C}$ be open and $a \in U$. Consider the ideal

$$I_a \coloneqq \{f \in C^\infty(U, \mathbb{C}) : f(a) = 0\} \subset C^\infty(U, \mathbb{C}).$$

The quotient vector space $T_a^*(U) \coloneqq I_a/I_a^2$ is called the **cotangent space of U at a.**

In particular, the cotangent space $T_a^*(U)$ is a $C^\infty(U, \mathbb{C})$-module (with pointwise multiplication). Also note that I_a is the kernel of the evaluation homomorphism $\mathrm{ev}_a \colon \mathbb{C}^\infty(U, \mathbb{C}) \to \mathbb{C}$.

Lemma 4.2. We have

$$I_a = (x \mapsto \Re(x-a), x \mapsto \Im(x-a))$$

and thus in particular

$$I_a^2 = \{f \in C^\infty(U, \mathbb{C}) : f(a) = 0, Df(a) = 0\}.$$

Proof. It is clear that the two generators are contained in I_a. For the other inclusion, let $f \colon U \to \mathbb{C} \cong \mathbb{R}^2$ be a smooth function and consider a ball $B_\epsilon(a) \subset U$. For $x \in B_\epsilon(a)$, we have

$$\partial_t f(a + t(x-a)) = Df(a + t(x-a)) \cdot (x-a) \in \mathbb{R}^2$$

and thus by the fundamental theorem of calculus

$$\begin{aligned} f(x) &= f(a) + \int_0^1 \partial_t f(a + t(x-a))dt = f(a) + \int_0^1 Df(a + t(x-a)) \cdot (x-a)dt \\ &= f(a) + (x_1 - a_1) \int_0^1 \partial_{x_1} f(a + t(x-a))dt + (x_2 - a_2) \int_0^1 \partial_{x_2} f(a + t(x-a))dt. \end{aligned}$$

Therefore, if $f \in I_a$, then the smooth functions $B_\epsilon(a) \to \mathbb{C}$

$$g_1(x) \coloneqq \int_0^1 \partial_{x_1} f(a + t(x-a))dt, \qquad g_2(x) \coloneqq \int_0^1 \partial_{x_2} f(a + t(x-a))dt$$

satisfy

$$f|_{B_\epsilon(a)} = \Re(x-a) \cdot g_1 + \Im(x-a) \cdot g_2.$$

By choosing a bump function $\psi \colon U \to \mathbb{R}$ that is 1 on $B_\delta(a) \subset B_\epsilon(a)$ for some $\delta \in (0, \epsilon)$ and 0 on the complement of $B_\epsilon(a)$, it follows that

$$f = \psi \cdot (\Re(x-a) \cdot g_1 + \Im(x-a) \cdot g_2) + (1 - \psi) \cdot f \in I_a.$$

Finally, the description of I_a^2 follows by noting that it is generated by

$$\left\{x \mapsto (\Re(x-a))^2, x \mapsto (\Im(x-a))^2, x \mapsto \Re(x-a) \cdot \Im(x-a)\right\}$$

and by using the product rule on a general element of I_a^2. □

This result generalizes in a straightforward way to general manifolds, see here.

As a consequence, I_a^2 consists of precisely those functions whose Taylor polynomial of first order vanishes. Therefore, the cotangent space of U at $a \in U$ can be thought of as the set of all “first order behaviors” at a; two functions $f, g \in I_a$ are equal in $T_a^*(U)$ if and only if their first order behaviors agree.

In particular, every smooth function $f\colon U \to \mathbb{C} \cong \mathbb{R}^2$ has a “first order behavior”, giving a map $C^\infty(U, \mathbb{C}) \to T_a^*(U)$. However, to ensure that $f \in I_a$, we must consider $f - f(a)$ instead of f.

Definition 4.3. For $U \subset \mathbb{C}$ open and $f \in C^\infty(U, \mathbb{C})$, its **differential at** $a \in U$ is

$$d_a f \coloneqq (f - f(a)) \bmod I_a^2 \in T_a^*(U),$$

giving a map

$$d_a\colon C^\infty(U, \mathbb{C}) \xrightarrow{\cdot - f(a)} I_a \xrightarrow{\bmod I_a^2} T_a^*(U),$$

which is just the canonical projection $I_a \twoheadrightarrow I_a/I_a^2$ on I_a.

By Taylor's theorem, $f \in C^\infty(U, \mathbb{C})$ can be written as

$$f(x) = f(a) + Df(a) \cdot (x - a) + \mathcal{O}((x - a)^2) \qquad (x \to a). \tag{2}$$

The next lemma shows that the rest term $\mathcal{O}((x - a)^2)$ is in I_a^2, so that the differential of f is just the “best linear approximation” of f; i.e. the function $x \mapsto Df(a) \cdot (x - a)$. In the theorem after that, this observation is then shown to be a $\mathbb{C}$-linear isomorphism.

Lemma 4.4. We have

$$C^\infty(U, \mathbb{C}) \cap \mathcal{O}\big((x - a)^2\big) = \left\{ f \in C^\infty(U, \mathbb{C}) : \lim_{x \to a} \frac{f(x)}{|x - a|^2} \text{ exists} \right\} \subset I_a^2 \qquad (x \to a).$$

Proof. Because $\lim_{x \to a} \frac{f(x)}{|x-a|^2} = \lim_{h \to 0} \frac{f(a+h)}{|h|^2}$ whenever one of those limits exists, this follows for $f \in C^\infty(U, \mathbb{C})$ (using Lemma 4.2) by observing that

$$f(a) = \lim_{h \to 0} \frac{f(a+h)}{|h|^2} \cdot |h|^2, \qquad \lim_{h \to 0} \frac{f(a+h)}{|h|} = \lim_{h \to 0} \frac{f(a+h)}{|h|^2} \cdot |h|. \qquad \square$$

Since Taylor's theorem (2) remains true for $f \in C^1(U, \mathbb{C})$, much of the theory easily generalizes to $C^1(U, \mathbb{C})$ instead of $C^\infty(U, \mathbb{C})$. Lec 6 2023-05-24

Theorem 4.5. For $U \subset \mathbb{C}$ open, the total derivative D induces a $\mathbb{C}$-linear isomorphism

$$D\colon T_a^*(U) \to \operatorname{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}),\ f \mapsto Df(a).$$

Here $\operatorname{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$ denotes the $\mathbb{C}$-vector space of $\mathbb{R}$-linear maps $\mathbb{C} \to \mathbb{C}$.

Proof. Consider the $\mathbb{R}$-linear surjective map

$$\psi\colon I_a \to \operatorname{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}),\ f \mapsto Df(a).$$

To see that it is also $\mathbb{C}$-linear, it suffices to show that $\psi(i \cdot f) = i \cdot \psi(f)$ for all $f \in I_a$, because $\{1, i\}$ is a $\mathbb{R}$-basis of $\mathbb{C}$:

$$\psi(i \cdot f) = D \begin{pmatrix} -f_2 \\ f_1 \end{pmatrix} (a) = D\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot f \right)(a) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot Df(a) = i \cdot \psi(f).$$

Because $\ker(\psi) = I_a^2$ by Lemma 4.2, the assertion follows. $\square$

Since this isomorphism is in particular $\mathbb{R}$-linear and $\mathbb{C}$ can be thought of as the tangent space $T_a(U)$ at a, this shows that the cotangent space $T_a^*(U)$ can also be viewed as the dual space (over $\mathbb{C}$) of the tangent space $T_a(U)$ (thus the notation). In fact, this result holds true for arbitrary manifolds, see here.

Definition 4.6. The **Wirtinger operators** (or **Wirtinger derivatives**) are the linear operators $C^1(U,\mathbb{C}) \to C^0(U,\mathbb{C})$

$$\partial_z := \frac{1}{2}(\partial_x - i\partial_y), \qquad \partial_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y).$$

Here ∂_x denotes the derivative operator w.r.t. the first component (when viewing $\mathbb{C}$ as $\mathbb{R}^2$) and ∂_y denotes the derivative operator w.r.t. the second component.

By the Cauchy-Riemann equations, the kernel of $\partial_{\bar{z}}$ are precisely the holomorphic functions $\ker(\partial_{\bar{z}}) = \mathcal{O}(U)$. Additionally, because $\partial_x(f) = f'$ (where f' denotes the complex derivative) and $\partial_y f = i \cdot f'$ for any holomorphic function $f \in \mathcal{O}(U)$, it follows that $\partial_z(f) = f'$.

Theorem 4.7. For an open subset $U \subset \mathbb{C}$ and $a \in U$, $\{d_a(\Re), d_a(\Im)\}$ is a basis of $T_a^*(U)$. We also denote $d_a(\Re)$ by $d_a x$ and $d_a(\Im)$ by $d_a y$, since it is common to write $z = x + iy$ with $x = \Re(z)$, $y = \Im(z)$.
The basis representation of $d_a f$ for $f \in C^\infty(U,\mathbb{C})$ (in particular for $f \in T_a^*(U)$ since then $d_a f = f$) is (where $\partial_x f(a), \partial_y f(a) \in \mathbb{C}$)

$$d_a f = \partial_x f(a) \cdot d_a x + \partial_y f(a) \cdot d_a y.$$

Similarly, $\{d_a \mathrm{id}_U, d_a \bar{\cdot}\}$ (the differential of the identity and complex conjugation; also written as $d_a z$ and $d_a \bar{z}$) is a basis of T_a^* and

$$d_a f = \partial_z f(a) \cdot d_a z + \partial_{\bar{z}} f(a) \cdot d_a \bar{z}.$$

Proof. This can be proven directly (using (2) for the representation of $d_a f$), but it is quicker to deduce it using the isomorphism from Theorem 4.5.
Because $d_a x$ corresponds to the map $\mathbb{R}^2 \to \mathbb{R}^2, (x,y) \mapsto (x - \Re(a), 0)$ (and similarly for $d_a y$), we have

$$D(d_a x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ D(d_a y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ i \cdot D(d_a x) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ i \cdot D(d_a y) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where we used that multiplication by i corresponds to left multiplication with the matrix representing a 90 degree anticlockwise rotation. Therefore, $D(d_a x)$ and $D(d_a y)$ form a $\mathbb{C}$-basis of $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$ and since $D\colon T_a^*(U) \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$ is a $\mathbb{C}$-linear isomorphism, $d_a x$ and $d_a y$ must be a basis of $T_a^*(U)$. Because the total derivative of f is the same as that of its differential $d_a f$ the first representation of $d_a f$ follows.
Similarly, we calculate

$$D(d_a z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ D(d_a \bar{z}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ i \cdot D(d_a z) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ i \cdot D(d_a \bar{z}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and solving the linear equation $Df = (a + ib) \cdot D(d_a z) + (c + id) \cdot D(d_a \bar{z})$ yields the assertion. $\square$

Definition 4.8. The **cotangent bundle** of an open set $U \subset \mathbb{C}$ is $T^*U = \coprod_{a \in U} T^*_a U$.

This set can be given a topology and a smooth structure that makes it into a manifold. With this, the canonical projection $\pi\colon T^*U \to U, (a,t) \mapsto a$ can be shown to be smooth (in particular continuous). However, we will skip this technical step.
In fact, the cotangent bundle of an arbitrary smooth manifold can be defined and endowed the structure of a smooth manifold.

4.2 Differential Forms

Definition 4.9. A **differential form of order** 0 (or **differential** 0**-form**) is a smooth function $U \to \mathbb{C}$.
A smooth function $\sigma\colon U \to T^*U$ that is right-inverse to the canonical projection $\pi\colon T^*U \to U$ is called a **differential form of order** 1. The set of differential forms of order k is denoted by $\Omega^k(U)$.

Explicitly, a differential form of order 1 is a smooth function $\sigma\colon U \to T^*U$ such that $\sigma(a) \in T^*_a(U)$ for all $a \in U$. Note that $\Omega^1(U)$ is a $C^\infty(U,\mathbb{C})$-module and in particular a $\mathbb{C}$-vector space (with pointwise addition and multiplication).

Example 4.10. Any $f \in C^\infty(U,\mathbb{C})$ induces a differential form of order 1 by its differential

$$df\colon U \to T^*(U),\ a \mapsto d_a f.$$

In particular, we have the differential 1-forms dx, dy, dz and $d\bar{z}$.

From the definition of these differential 1-forms and the fact that $d_a\colon C^\infty(U,\mathbb{C}) \to T^*_a(U)$ is $\mathbb{C}$-linear, we have

$$dz = dx + idy, \quad d\bar{z} = dx - idy, \quad dx = \frac{dz + d\bar{z}}{2}, \quad dy = \frac{d\bar{z} - dz}{2} i. \tag{3}$$

We can now reinterpret Theorem 4.7.

Theorem 4.11 (**Bases of** 1**-forms**)**.**
As a module over $C^\infty(U,\mathbb{C})$, the differential 1-forms $\Omega^1(U)$ are two dimensional with one basis given by $\{dx, dy\}$ and another one given by $\{dz, d\bar{z}\}$.
Explicitly, any differential 1-form ω can be written as

$$\omega = \partial_x \cdot dx + \partial_y \cdot dy = \partial_z \cdot dz + \partial_{\bar{z}} \cdot d\bar{z}.$$

Any 1-form $\omega\colon U \to T^*(U)$ is thus of the form

$$\omega(a) = \partial_x(\omega(a))(a)\cdot d_a x + \partial_y(\omega(a))(a)\cdot d_a y = \partial_z(\omega(a))(a)\cdot d_a z + \partial_{\bar{z}}(\omega(a))(a)\cdot d_a \bar{z} \quad \forall\ a \in U.$$

Definition 4.12. Let V be a vector space over a field $\mathbb{K}$. The **second exterior power** is the quotient vector space

$$\Lambda^2 V = (V \otimes V)/\langle v \otimes v : v \in V\rangle.$$

The second exterior power is actually a subspace of the *exterior algebra* (or *Grassman algebra*), see wikipedia.
The tensor product $\otimes\colon V \to V \to \Lambda^2 V$ in this context is also called the *exterior product* and has some important properties, which we now specify without proof. We will make use of these properties instead of the explicit definiton.

Lemma 4.13. Let V be a vector space over an arbitrary field. There exists an operation

$$\wedge : V \times V \to \Lambda^2 V,$$

called the **exterior product** having the following two properties (for $v_1, v_2, v_3 \in V$, $\lambda \in \mathbb{C}$):

(a) *bilinear*:

$$(v_1 + v_2) \wedge v_3 = v_1 \wedge v_3 + v_2 \wedge v_3, \qquad (\lambda v_1) \wedge v_2 = \lambda \cdot v_1 \wedge v_2$$

and the linerity in the second argument follows with the second property;

(b) *anticommutative*: $v_1 \wedge v_2 = -v_2 \wedge v_1$.

Furthermore, if V is finite dimensional with basis $\{e_1, \dots, e_n\}$, then $\Lambda^2 V$ has the basis $\{e_i \wedge e_j : i < j, i, j \in \{1, \dots, n\}\}$ and thus has dimension $\binom{n}{2}$.

Applying this construction to the cotangent space $T_a^*(U)$ of U at a gives rise to the differential 2-forms. In the following definition, we will again omit specifying the smooth structure on the disjoint union.

Definition 4.14. Let $U \subset \mathbb{C}$ be open. A **differential form of order** 2 on U is a smooth function

$$\omega \colon U \to \coprod_{a \in U} \Lambda^2(T_a^*(U))$$

that is right-inverse to the canonical projection $\pi \colon \coprod_{a \in U} \Lambda^2(T_a^*(U)) \to U$ (i.e. it satisfies $\omega(a) \in \Lambda^2(T_a^*(U))$ for all $a \in U$).

We saw in Theorem 4.7 that each cotangent space $T_a^*(U)$ has the $\mathbb{C}$-bases $\{d_a x, d_a y\}$ and $\{d_a z, d_a \bar{z}\}$. By Lemma 4.13, this implies that $\Lambda^2(T_a^*(U))$ has the two $\mathbb{C}$-bases $\{d_a x \wedge d_a y\}$ and $\{d_a z \wedge d_a \bar{z}\}$. Therefore, any differential 2-form $\omega \colon U \to \coprod_{a \in U} \Lambda^2(T_a^*(U))$ can be written as $\omega = f \cdot dx \wedge dy$ and $\omega = h \cdot dz \wedge d\bar{z}$ with unique functions $f, h \colon U \to \mathbb{C}$, which can be shown to be smooth. We summarize this result.

Theorem 4.15 (Bases of 2-forms).
Let $U \subset \mathbb{C}$ be open. As a module over $C^\infty(U, \mathbb{C})$, the differential 2-forms $\Omega^2(U)$ are one dimensional with one basis given by $\{dx \wedge dy\}$ and another one by $\{dz \wedge d\bar{z}\}$. Thus any differential 2-form ω can be written uniquely as

$$\omega = f \cdot dz \wedge d\bar{z} = h \cdot dx \wedge dy,$$

where $f, h \in C^\infty(U, \mathbb{C})$.

Definition 4.16. For $U \subset \mathbb{C}$ open and $k \in \mathbb{N}$, the linear map

$$d \colon \Omega^1(U) \to \Omega^2(U), \;\; f \cdot dz + g \cdot d\bar{z} \mapsto df \wedge dz + dg \wedge d\bar{z}$$

is called the **exterior derivative**.

With the formulas

$$df = \partial_z f \cdot dz + \partial_{\bar{z}} f \cdot d\bar{z}, \qquad dg = \partial_z g \cdot dz + \partial_{\bar{z}} g \cdot d\bar{z}$$

from Theorem 4.11, the exterior derivative is equivalently given by

$$d(f \cdot dz + g \cdot d\bar{z}) = (\partial_z g - \partial_{\bar{z}} f) \cdot dz \wedge d\bar{z}. \tag{4}$$

It is similarly straightforward to plug in (3) and obtain the equivalent definition

$$d(f \cdot dx + g \cdot dy) = (\partial_x g - \partial_y f) dx \wedge dy.$$

Lec 7
2023-05-31

By Theorem 4.11, a 1-form $\omega \in \Omega^1(U)$ can be written as $\omega(a) = \partial_z(\omega(a))(a) \cdot d_a z + \partial_{\bar{z}}(\omega(a))(a) \cdot d_a \bar{z}$ (for $a \in U$) and we also have $\Omega^1(U) = \langle dz \rangle \oplus \langle d\bar{z} \rangle$ (as $C^\infty(U, \mathbb{C})$-modules). The observation $\ker(\partial_{\bar{z}}) = \mathcal{O}(U)$ (by the Cauchy-Riemann equations) thus inspires the following definition.

Definition 4.17. A 1-form $\omega \in \Omega^1(U)$ is called **holomorphic**, if $\omega = f \cdot dz$ with $f \in \mathcal{O}(U)$. It is called **closed**, if $d\omega = 0$.
The $\mathbb{R}$-vector space of all closed 1-forms on U is denoted by $Z^1(U)$. It is the kernel of the linear map $d\colon \Omega^1(U) \to \Omega^2(U)$.

Proposition 4.18. Let $U \subset \mathbb{C}$ be open. A 1-form $\omega = f \cdot dz \in \Omega^1(U)$ with $f \in C^\infty(U, \mathbb{C})$ is closed if and only if it is holomorphic.

Proof. This follows immediately from (4) as $d\omega = -\partial_{\bar{z}} f \cdot dz \wedge d\bar{z}$ and $\ker(\partial_{\bar{z}}) = \mathcal{O}(U)$. □

We now briefly introduce 0-chains and 1-chains, which originally come from singular homology, a tool used in algebraic topology to characterize how many holes a topological space has.

Definition 4.19. Let $U \subset \mathbb{C}$ be an open set. The free abelian group $C_1(U)$ generated by the set

$$X := \{\gamma\colon [0,1] \to U \text{ continuously differentiable curve}\}$$

is called the **group of 1-chains**. The subgroup generated by the closed curves in X (i.e. $\gamma \in X$ with $\gamma(0) = \gamma(1)$) is called the **group of 1-cycles** and is denoted by $Z_1(U)$. The free abelian group $C_0(U)$ generated by U is called the **group of 0-chains**.

Explicitly, an element of the group of 1-chains can be written as $\gamma = \sum_{j=1}^k \alpha_j \gamma_j$ for $\alpha_j \in \mathbb{Z}$ and $\gamma_j \in X$.

4.3 The Cauchy Integral Theorem for differential 1-Forms

In general manifold theory, differential n-forms serve as the "objects" that can be integrated over. For us, it suffices to integrate over 1-forms.

Definition 4.20. Let $U \subset \mathbb{C}$ be open, $\omega \in \Omega^1(U)$ a 1-form and $\gamma\colon [0,1] \to \mathbb{C}$ a continuously differentiable path. The **integral** of $\omega = f \cdot dx + g \cdot dy$ over γ is defined to be

$$\int_\gamma \omega := \int_0^1 (f \circ \gamma)(t) \cdot (\Re \circ \gamma)'(t) dt + \int_0^1 (g \circ \gamma)(t) \cdot (\Im \circ \gamma)'(t) dt \in \mathbb{C}.$$

If $\gamma = \sum_{j=1}^k \alpha_j \gamma_j \in C_1(U)$ is instead a 1-chain, then we set

$$\int_\gamma \omega := \sum_{j=1}^k \alpha_j \int_{\gamma_j} \omega.$$

Let us first notice that for holomorphic 1-forms, this generalizes the complex path integral in the following sense.

Lemma 4.21 (Integral of 1-forms generalizes complex path integral).
For $U \subset \mathbb{C}$ be open, $\omega = fdz \in \Omega^1(U)$ a holomorphic 1-form and $\gamma \in C^1(U)$ a 1-chain, we have $\int_\gamma \omega = \int_\gamma f dz$.

Proof. It suffices to prove this for a continuously differentiable path $\gamma\colon [0,1] \to U$ and indeed, using the fact that $dz = dx + i \cdot dy$, we calculate

$$\begin{aligned}\int_\gamma \omega &= \int_0^1 (f \circ \gamma)(t) \cdot (\Re \circ \gamma)'(t)dt + \int_0^1 (f \circ \gamma)(t) \cdot i(\Im \circ \gamma)'(t)dt \\ &= \int_0^1 (f \circ \gamma)(t) \cdot \gamma'(t)dt = \int_\gamma f dz.\end{aligned}$$ □

This integral operator depends on ω and γ, we can thus view it as a function

$$\int : C_1(U) \times \Omega^1(U) \to \mathbb{R},\ (\gamma, \omega) \mapsto \int_\gamma \omega,$$

which is a group homomorphism ($\mathbb{Z}$-linear) in the first component and $\mathbb{R}$-linear in its second component. We restrict this function to $Z_1(U) \times Z^1(U)$, where $Z_1(U)$ is the group of 1-cycles and $Z^1(U)$ is the $\mathbb{R}$-vector space of closed 1-forms.
Note that we may equivalently view this function as a group homomorphism

$$Z_1(U) \to \big\{Z^1(U) \to \mathbb{R} \text{ linear}\big\}.$$

Definition 4.22. Let $U \subset \mathbb{C}$ be an open set and consider the kernel of the previous group homomorphism

$$B_1(U) := \left\{\gamma \in Z_1(U) \colon \int_\gamma \omega = 0\ \forall\, \omega \in \Omega^1(U) \text{ closed}\right\} \subset Z_1(U),$$

whose elements are called **null homologous**.
The quotient group $H_1(U) := Z_1(U)/B_1(U)$ is called the **first homology group of** U.
Two cycles $\gamma, \gamma' \in Z_1$ are called **homologous relative to** U, if they are equal in $H_1(U)$; i.e. if $\int_\gamma \omega = \int_{\gamma'} \omega$ for all closed $\omega \in \Omega^1(U, \mathbb{C})$.

By instead viewing the integral operator as a $\mathbb{R}$-linear map

$$\Omega^1(U) \to \{Z_1(U) \to \mathbb{R} \text{ group homomorphism}\},$$

we obtain the following "dual" definition.

Definition 4.23. For $U \subset \mathbb{C}$ open, a 1-form $\omega \in \Omega^1(U)$ is called **null cohomologous**, if it is in the kernel of the previous $\mathbb{R}$-linear map; that is, if $\int_\gamma \omega = 0$ for all 1-cycles $\gamma \in Z_1(U)$.

Putting all this terminology together and recalling that for holomorphic 1-forms, the integral of 1-forms generalizes the complex path integral (see Lemma 4.21), we can state a "modern version" of Cauchy's integral theorem.

Theorem 4.24 (Cauchy's Integral Theorem). Let $U \subset \mathbb{C}$ be open and simply connected. Then every holomorphic 1-form $\omega \in \Omega^1(U)$ is null cohomologous.

5 Holomorphic Functions with prescribed Zeros and Poles

5.1 Infinite Products

In this section, we define and study infinite products of complex numbers, following [JS87, 3.8].

Definition 5.1. Let $(b_j)_{j\in\mathbb{N}} \in \mathbb{C}$ be a sequence and $p_n := \prod_{j=0}^n b_j$. The sequence (p_n) is called an **infinite product**. If $\lim_{n\to\infty} p_n$ exists (in $\mathbb{C}$) and is not 0, then we call the infinite product **convergent** and write the limit as $\prod_{j=0}^\infty b_j$. Furthermore, if there exists $N \in \mathbb{N}$, such that $b_j = 0$ for some $j < N$ and $b_j \neq 0$ for all $n \geq N$ and $\prod_{j=N}^\infty b_j$ converges (as defined above), then we say that the infinite product **converges to** 0 and write $\prod_{j=0}^\infty b_j = 0$.
If $p_n \to 0$ and no factor is zero or the limit is infinite, then the infinite product is called **divergent**.

One natural question is to ask when an infinite product converges. For simplicity, we will here and in the following write $b_j = 1 + c_j$. Since $p_{n+1} = p_n \cdot b_{n+1} = p_n + p_n \cdot c_{n+1}$, a necessary condition for convergence of an infinite product is $\lim_{j\to\infty} c_j = 0$ (or equivalently $\lim_{j\to\infty} b_j = 1$).

One can use the logarithm in order to convert infinite products to infinite sums. Indeed, the main motivation for the previous somewhat artificial definition is to make the following theorem particularly clean.

Theorem 5.2. The product $\prod_{j=0}^\infty b_j$ converges if and only if there exists $N \in \mathbb{N}$, such that the sum $\sum_{j=N}^\infty \mathrm{Log}(b_j)$ is well-defined and converges.

Proof. If infinitely many of the b_j lie in $(-\infty, 0] \subset \mathbb{C}$, the product does not converge and the sum is not well-defined for any $N \in \mathbb{N}$. If only finitely many of the factors lie in $(-\infty, 0] \subset \mathbb{C}$, they can be removed by choosing the "starting index" $N \in \mathbb{N}$ large enough. Therefore, we may assume that $b_j \in \mathbb{C}^-$ for all $j \in \mathbb{N}$, so that the sum is well-defined for $N = 0$.
If the partial sums $s_n := \sum_{j=0}^n \mathrm{Log}(b_j)$ converge to $L \in \mathbb{C}$, then by continuity of exp, we have $\prod_{j=0}^n b_j = \exp(s_n) \to \exp(L) \neq 0$.
On the other hand, assume that $\prod_{j=0}^\infty b_j$ converges to $L \neq 0$ and write $p_n := \prod_{j=0}^n b_j$ as well as $s_n := \sum_{j=0}^n \mathrm{Log}(b_j)$. By potentially replacing b_0 with $-b_0$, we may assume that $L \in \mathbb{C}^-$. By Lemma 3.12, there exist $q_n \in \mathbb{Z}$, such that

$$s_n = \mathrm{Log}(p_n) + 2\pi i q_n$$

and we thus have to show that q_n is constant for sufficiently large $n \in \mathbb{N}$.
We calculate

$$\begin{aligned}
&2\pi i(q_{n+1} - q_n) \\
&= s_{n+1} - s_n + \mathrm{Log}(p_n) - \mathrm{Log}(p_{n+1}) \\
&= \mathrm{Log}(b_{n+1}) + \mathrm{Log}(p_n) - \mathrm{Log}(p_{n+1}) \\
&= \log(|b_{n+1}|) + \log(|p_n|) - \log(|p_{n+1}|) + i(\mathrm{Arg}(b_{n+1}) + \mathrm{Arg}(p_n) - \mathrm{Arg}(p_{n+1}))
\end{aligned}$$

and by comparing imaginary parts, we see that

$$\begin{aligned}|q_{n+1} - q_n| &= \frac{1}{2\pi}|\mathrm{Arg}(b_{n+1}) + \mathrm{Arg}(p_n) - \mathrm{Arg}(p_{n+1})| \\ &\le \frac{1}{2\pi}(|\mathrm{Arg}(b_{n+1})| + |\mathrm{Arg}(p_n) - \mathrm{Arg}(p_{n+1})|).\end{aligned}$$

Since $b_n \to 1$ and (p_n) is a Cauchy sequence that for large enough $n \in \mathbb{N}$ lies in $\mathbb{C}^-$ (where $\mathrm{Arg}\colon \mathbb{C}^\times \to \mathbb{R}$ is continuous), we conclude that $|q_{n+1} - q_n| < 1$ for sufficiently large n. Because it is also a whole number, this implies $q_{n+1} = q_n$, which was to be shown. □

With this theorem, we can translate the notion of absolute convergence from sums to products.

Definition 5.3. An infinite product $\prod_{j=0}^\infty b_j$ is called **absolutely convergent**, if the infinite sum $\sum_{j=0}^\infty \log(b_j)$ converges absolutely.

In particular, any infinite product that converges absolutely also converges, just like for infinite sums.

Theorem 5.4. An infinite product $\prod_{j=0}^\infty (1 + c_j)$ converges absolutely if and only if the infinite sum $\sum_{j=0}^\infty c_j$ converges absolutely.

Proof. The two sums $\sum_{j=0}^\infty \mathrm{Log}(1 + c_j)$ and $\sum_{j=0}^\infty c_j$ can converge only if $c_j \to 0$. In particular, for all sufficiently large $j \in \mathbb{N}$, we have $|c_j| < \frac{1}{2}$. For $|z| < 1$, the Taylor series of Log at 1 is

$$\mathrm{Log}(1 + z) = \sum_{k=1}^\infty (-1)^{k+1}\frac{z^k}{k},$$

so for sufficiently large $j \in \mathbb{N}$, we calculate

$$|\mathrm{Log}(1 + c_j) - c_j| = \left|\sum_{k=2}^\infty (-1)^{k+1}\frac{c_j^k}{k}\right| \le \frac{1}{2}\sum_{k=2}^\infty |c_j|^k = \frac{|c_j|^2}{2(1 - |c_j|)} \le \frac{1}{2}|c_j|,$$

which means that $\mathrm{Log}(1 + c_j) \in B_{\frac{1}{2}|c_j|}(c_j)$ and therefore

$$\frac{1}{2}|c_j| \le |\mathrm{Log}(1 + c_j)| \le \frac{3}{2}|c_j|.$$ □

As a consequence, the infinite product $\prod_{j=0}^\infty (1 + c_j)$ converges absolutely if $\prod_{j=0}^\infty (1 + |c_j|)$ converges.
Furthermore, while the theorem implies that $\prod_{j=0}^\infty (1 + c_j)$ converges if the infinite sum $\sum_{j=0}^\infty c_j$ converges absolutely, it can happen that the infinite product converges, but the infinite sum does not. We give an example of this phenomena.

Example 5.5. Consider the sequence

$$c_{2j-1} = \frac{1}{\sqrt{j}}, \qquad c_{2j} = -\frac{1}{\sqrt{j}} + \frac{1}{j} \qquad \forall\, j \in \mathbb{N}_{>0}.$$

The even partial sums $s_{2n} := \sum_{j=1}^{2n} c_j$ satisfy $s_{2n} = \sum_{j=1}^n \frac{1}{j}$ and thus the infinite sum $\sum_{j=0}^\infty c_j$ does not converge. However, the calculation

$$(1 + c_{2j-1})(1 + c_{2j}) = \left(1 + \frac{1}{\sqrt{j}}\right)\left(1 - \frac{1}{\sqrt{j}}\right) + \frac{1}{j}\left(1 + \frac{1}{\sqrt{j}}\right) = 1 + j^{-\frac{3}{2}}$$

shows that the n-th partial product $p_n := \prod_{j=1}^{n}(1+c_j)$ satisfies

$$p_{2k} = \prod_{j=1}^{k}\Big(1+j^{-\frac{3}{2}}\Big), \qquad p_{2k-1} = \Big(1+\frac{1}{\sqrt{k}}\Big)\cdot p_{2(k-1)} = \Big(1+\frac{1}{\sqrt{k}}\Big)\cdot\prod_{j=1}^{k-1}\Big(1+j^{-\frac{3}{2}}\Big).$$

Because the infinite sum $\sum_{j=1}^{\infty} j^{-\frac{3}{2}}$ converges absolutely and $1+\frac{1}{\sqrt{k}} \to 1$, we conclude (using the previous theorem) that $\prod_{j=1}^{\infty}(1+c_j)$ converges (but not absolutely).

5.2 The Mittag-Leffler Theorem

Lec 8 2023-06-07

Using partial fraction decomposition (see wikipedia), every rational function $R\colon \mathbb{C}\to\mathbb{C}$ can be written in the form

$$R(z) = \sum_{j=0}^{r}\sum_{k=1}^{K_j}\frac{a_{j,k}}{(z-z_j)^k} + \sum_{j=0}^{s} b_j z^j \qquad \text{for } a_{j,k}, b_j\in\mathbb{C}; K_j, r, s\in\mathbb{N}. \tag{5}$$

We ask whether every meromorphic function $f\in\mathcal{M}(\mathbb{C})$ can be represented as a sum of a holomorphic function and principal parts (the double sum in (5). On the other hand, given a countably infinite number of points in $\mathbb{C}$, we want to construct $f\in\mathcal{M}(\mathbb{C})$, such that its poles are precisely those points. Of course, if those points have an accumulation point, then no such f can exist (by Theorem 1.12). Furthermore, given finitely many points $\{z_1,\dots,z_r\}\subset\mathbb{C}$, such a meromorphic function is

$$f(z) = \sum_{j=1}^{r}\sum_{k=1}^{K_j}\frac{a_{j,k}}{(z-z_j)^k} + g(z) \qquad \text{for } g\in\mathcal{O}(\mathbb{C}), a_{j,k}, b_j\in\mathbb{C}; K_j, r\in\mathbb{N}_{>0}.$$

The case of countably infinitely many poles is the content of the following theorem.

Theorem 5.6 (**Mittag-Leffler Theorem (special case)**).
Let $(z_j)\in\mathbb{C}$ be a sequence, such that $|z_j|<|z_{j+1}|$ for all $j\in\mathbb{N}$ and $|z_j|\to\infty$. Furthermore, for every $j\in\mathbb{N}$, we have coefficients $a_{j,k}$, such that we may construct the principal parts

$$h_j(z) := \sum_{k=1}^{K_j}\frac{a_{j,k}}{(z-z_j)^k}.$$

Then there exists $f\in\mathcal{M}(\mathbb{C})$, such that its set of poles is $\{z_j : j\in\mathbb{N}\}$ and that its principal part at every z_j is h_j. Additionally, two such meromorphic functions are unique up to addition of a holomorphic function on $\mathbb{C}$.

Proof. If $z_0=0$ and we have constructed f such that it has the desired properties except at z_0, then the function $f+h_0$ has the desired properties also at z_0. Therefore, we may assume that $z_0\neq 0$.
Choose a summable sequence $(\epsilon_j)\in\mathbb{R}_+$ and a strictly increasing sequence $(r_j)\in\mathbb{R}_+$ with $r_j\to\infty$ and $r_j<|z_j|$ for all $j\in\mathbb{N}$. Because all h_j are holomorphic on $B_{|z_0|}(0)$, they can be represented as a power series $h_j(z)=\sum_{n=0}^{\infty} b_{j,n}z^n$ (with $b_{j,n}\in\mathbb{C}$) on $B_{|z_0|}(0)$. Furthermore, each h_j converges uniformly on the disk $B_{r_j}(0)\subset B_{|z_j|}(0)$. Therefore, for every $j\in\mathbb{N}$, we can choose an index $m_j\in\mathbb{N}$, such that

$$g_j(z) := \sum_{n=0}^{m_j} b_{j,n}z^n \in \mathcal{O}(\mathbb{C})$$

satisfies $|h_j(z) - g_j(z)| < \epsilon_j$ for all $z \in B_{r_j}(0)$.
We claim that $f := \sum_{j=0}^{\infty}(h_j - g_j)$ converges on $\mathbb{C} \setminus \{z_j : j \in \mathbb{N}\}$ and is the desired meromorphic function.
Let $r > 0$ be given and pick $N \in \mathbb{N}$ large enough such that $r_N \geq r$. Denoting the N-th tail sum of f by $R := \sum_{j=N}^{\infty}(h_j - g_j)$, for $z \in B_r(0)$ we observe that

$$\sum_{j=N}^{\infty}|(h_j - g_j)(z)| < \sum_{j=N}^{\infty}\epsilon_j \to 0 \quad (N \to \infty),$$

so R is absolutely and uniformly convergent on $B_r(0)$. As a uniform limit of holomorphic functions, R is holomorphic on $B_r(0)$. The corresponding partial sum $S := \sum_{j=0}^{N-1}(h_j - g_j)$ is holomorphic on $B_r(0) \setminus \{z_0, \ldots, z_{N-1}\}$. For $z_j \in B_r(0)$, we have $j < N$ and $S - h_k$ is holomorphic at z_k, which shows that $f = S + R$ has the desired properties.
Finally, the difference of two such meromorphic functions is holomorphic, since the principal parts at the z_j cancel. □

The result is easily generalized to the case where $\{z_j : j \in \mathbb{N}\}$ is discrete and closed (or equivalently has no accumulation point; see Lemma 1.11) and there exists a fixed $M \in \mathbb{N}$, such that for any $k \in \mathbb{N}$, at most M of the z_j satisfy $|z_j| = |z_k|$: For $i \in \{1, \ldots, M\}$, choose sets U_i, such that $\coprod_{i=1}^{n} U_i = \{z_j : j \in \mathbb{N}\}$ and that each U_i is either finite or satisfies $|z_j| < |z_{j+1}|$ for all $j \in \mathbb{N}$. Then the theorem (or the trivial solution using rational functions) can be applied to each U_i and the results multiplied.
In fact, it can be shown that it is sufficient to assume that the z_j are discrete and closed.

Theorem 5.7 (**Mittag-Leffler Theorem**).
Theorem 5.6 remains true if the (z_j) form a discrete and closed set.

With the terminology of *principal part distributions*, the theorem can be stated more concisely.

Definition 5.8. Let $G \subset \mathbb{C}$ be a region $\{z_j \in G : j \in \mathbb{N}\}$ a discrete and closed set and $h^* \in \mathbb{C}[z]$ a polynomial with $h^*(0) = 0$. Defining the functions

$$h_j \colon \mathbb{C} \setminus \{z_j\} \to \mathbb{C},\ z \mapsto h^*\left(\frac{1}{z - z_j}\right),$$

the set $H := \{(z_j, h_j) : j \in \mathbb{N}\}$ is called a **principal part distribution**.

Every $f \in \mathcal{M}(G) \setminus \{0\}$ defines a principal part distribution $H(f)$ via its principal parts at its poles. This allows a restatement of Theorem 5.7.

Theorem 5.9 (**Mittag-Leffler Theorem for Principal Part Distributions**).
For every principal part distribution H on $\mathbb{C}$, there exists $f \in \mathcal{M}(\mathbb{C}) \setminus \{0\}$, such that $H = H(f)$.

5.3 The Weierstrass Factorization Theorem

Given finite collections $\alpha := \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{C}$ and $\nu := \{\nu_1, \ldots, \nu_n\} \subset \mathbb{N}_{>0}$, there exists a polynomial having α as its zeros with multiplicities ν (i.e. every α_j is a zero of multiplicity ν_j and there are no other zeros), namely

$$p(z) = \prod_{j=1}^{n}(z - \alpha_j)^{\nu_j}.$$

We ask whether a similar construction exists if α and ν are countably infinite sets.

Definition 5.10. For an open set $U \subset \mathbb{C}$, the set of nonvanishing holomorphic functions on U is denoted by $\mathcal{O}^*(U)$.

Note that $\mathcal{O}^*(U)$ is the group of units of $\mathcal{O}(U)$, so it is an abelian group w.r.t. pointwise multiplication.

Lemma 5.11. Let $G \subset \mathbb{C}$ be a simply connected region and $f \in \mathcal{O}^*(G)$. Then there exists $g \in \mathcal{O}(G)$, such that $f = \exp \circ g$.
g is called **logarithm of f on G** and is sometimes denoted by $\log(f)$.

Proof. Because $\frac{f'}{f} \in \mathcal{O}(G)$ and G is a simply connected region, it admits an antiderivative $g \in \mathcal{O}(G)$. Since

$$\left(\frac{\exp \circ g}{f}\right)' = \frac{\exp \circ g \cdot g' \cdot f - \exp \circ g \cdot f'}{f^2} = \frac{\exp \circ g}{f^2}\left(\frac{f'}{f} f - f'\right) = 0,$$

$\frac{\exp \circ g}{f}$ must be constant. Fixing an arbitrary $z_0 \in G$, there exists $C \in \mathbb{C}$, such that $\exp(C) = \frac{f(z_0)}{\exp(g(z_0))}$. Thus replacing g by $g + C \in \mathcal{O}(G)$ yields $\frac{(\exp \circ g)(z_0)}{f(z_0)} = 1$ and thus $\frac{\exp \circ g}{f} = 1$. □

It follows that a general solution for finite α and ν is of the form $p(z) \cdot \exp(g(z))$ for $g \in \mathcal{O}(\mathbb{C})$.

The case of countable infinite α and ν is settled by the following theorem.

Theorem 5.12 (Weierstrass Factorization Theorem (special case)).
Let $(\alpha_j) \in \mathbb{C}$ and $(\nu_j) \in \mathbb{N}_{>0}$ be two sequences satisfying $\alpha_0 = 0$, $|\alpha_j| < |\alpha_{j+1}|$ for all $j \in \mathbb{N}$ and $\alpha_j \to \infty$. Then there exists a function $f_0 \in \mathcal{O}(\mathbb{C})$ whose zeros are precisely the α_j and each α_j is a zero of multiplicity ν_j. Explicitly, f_0 is of the form

$$f_0(z) = z^{\nu_0} \cdot \prod_{j=1}^{\infty} \left(\left(1 - \frac{z}{\alpha_j}\right) \cdot \exp(p_j(z)) \right)^{\nu_j},$$

where each p_j is a polynomial given by a partial sum of the power series

$$-\log\left(1 - \frac{z}{\alpha_j}\right) = \sum_{l=1}^{\infty} \frac{1}{l}\left(\frac{z}{\alpha_j}\right)^l.$$

Furthermore, every function with these properties can be written as $f_0 \cdot \exp \circ g$ for $g \in \mathcal{O}(\mathbb{C})$.

Lec 9
2023-06-14

Proof. For fixed $j \in \mathbb{N}$, consider a function f that is locally near z_j of the form

$$f(z) := (z - \alpha_j)^{\nu_j} \tilde{f}(z) \qquad \text{for } \tilde{f} \text{ holomorphic, } \tilde{f}(z_j) \neq 0.$$

Its logarithmic derivative

$$(\log \circ f)'(z) = \frac{f'(z)}{f(z)} = \frac{\nu_j}{z - \alpha_j} + \frac{\tilde{f}'(z)}{\tilde{f}(z)}$$

has a pole of order 1 at α_j with principal part $\frac{\nu_j}{z-\alpha_j}$ (and residue ν_j).
Let $j > 0$. Because the desired function f_0 must have a zero at α_j with multiplicity ν_j, its

logarithmic derivative $g := \frac{f_0'}{f_0}$ has a pole of order 1 at α_j with principal part $g_j(z) := \frac{\nu_j}{z-\alpha_j}$. Using the geometric series, we have the power series representation

$$g_j(z) = \frac{\nu_j}{\alpha_j} \cdot \frac{1}{\frac{z}{\alpha_j} - 1} = -\nu_j \sum_{k=1}^{\infty} \frac{z^{k-1}}{\alpha_j^k} \qquad \text{for } z \in B_{|\alpha_j|}(0).$$

We now argue just like in the proof of Theorem 5.6. Let $(\epsilon_j) \in \mathbb{R}_+$ be a summable sequence and $(r_j) \in \mathbb{R}_+$ a strictly increasing sequence with $r_j \to \infty$ and $r_j < |z_j|$ for all $j \in \mathbb{N}$. Because the sum representation of g_j converges uniformly on $B_{r_j}(0)$, we can choose $m_j \in \mathbb{N}$, such that the partial sum $h_j(z) := -\nu_j \sum_{k=1}^{m_j} \frac{z^{k-1}}{\alpha_j^k}$, satisfies $|g_j(z) - h_j(z)| < \epsilon_j$ for all $z \in B_{r_j}(0)$. For given $r > 0$, we choose $N \in \mathbb{N}$ large enough so that $r_N \geq r$. Then we have $\sum_{j=N}^{\infty} |g_j - h_j| < \sum_{j=N}^{\infty} \epsilon_j$ for $z \in B_r(0)$ and thus the infinite sum $\sum_{j=1}^{\infty}(g_j - h_j)$ converges uniformly on $B_r(0) \setminus \{\alpha_j : j \in \mathbb{N}_{>0}\}$. Since $r > 0$ was arbitrary, it follows that $g(z) := \sum_{j=1}^{\infty}(g_j - h_j)$ converges uniformly on any bounded set that does not contain any α_j for $j > 0$.
The function

$$u_j(z) := \left(\left(1 - \frac{z}{\alpha_j}\right) \exp\left(\sum_{l=1}^{m_j} \frac{1}{l}\left(\frac{z}{\alpha_j}\right)^l\right)\right)^{\nu_j}$$

has logarithmic derivative $g_j - h_j$ and thus by the above is a reasonable approach. It remains to show that the infinite product

$$f_0(z) := z^{\nu_0} \prod_{j=1}^{\infty} u_j(z)$$

converges. To see this, let $R > 0$ be given and choose $N \in \mathbb{N}$ large enough, such that $|\alpha_j| > R$ for all $j \geq N$. By the Cauchy Integral theorem, an antiderivative of $g_j - h_j$ is

$$v_j(z) := \int_0^z \frac{u_j'(\xi)}{u_j(\xi)} d\xi = \int_0^z (g_j(\xi) - h_j(\xi)) d\xi$$

and the integral is independent of the chosen path in $B_R(0)$. The uniform convergence of $\sum_{j=N}^{\infty}(g_j - h_j)$ on $B_R(0)$ implies that $\sum_{j=N}^{\infty} v_j$ converges as well. Furthermore, since $\exp \circ v_j = \exp \circ \operatorname{Log} \circ v_j = u_j$, the previous sum is equal to $\sum_{j=N}^{\infty} \operatorname{Log}(u_j)$ and thus Theorem 5.2 shows that the desired product $\prod_{j=N}^{\infty} u_j$ converges on $B_R(0)$.
Finally, if $f_1 \in \mathcal{O}(\mathbb{C})$ is another function with the desired properties, then $\frac{f_1}{f_0} \in \mathcal{O}^*(\mathbb{C})$, so $\frac{f_1}{f_o} = \exp \circ g$ for some $g \in \mathcal{O}(\mathbb{C})$ by Lemma 5.11. □

This proof follows [FL94, Thm 2.2].
Just like for Theorem 5.6, there is a straightforward generalization of the theorem to the case that there is a uniform upper bound on the number of α_j that have the same absolute value and it can be shown that it suffices to assume that the α_j are discrete and closed (or equivalently have no accumulation point; see Lemma 1.11).

Theorem 5.13 (Weierstrass Factorization Theorem).
Theorem 5.12 remains true if the α_j are discrete and closed (though f_0 may have a different explicit form).

We aim to restate the Weierstrass Factorization theorem (Theorem 5.13) using the terminology of *divisors*, which we now introduce. A good reference (in german) is [RS07, 3.1.1].

Definition 5.14. Let $G \subset \mathbb{C}$ be a region. A map $D\colon G \to \mathbb{Z}$ is called a **divisor on** G, if the set $\{z \in G : D(z) \neq 0\}$ is discrete and closed in G.

By Lemma 1.11, the condition is equivalent to asking that the set has no accumulation points in G.

Lemma 5.15. $D\colon G \to \mathbb{Z}$ is a divisor if and only if for every compact subset $K \subset G$, there exist only finitely many $z \in K$ with $D(z) \neq 0$.

Proof. If a compact subset $K \subset G$ contains infinitely many $z \in K$ with $D(z) \neq 0$, then they have an accumulation point.
On the other hand, suppose that every compact subset $K \subset G$ contains only finitely many $z \in K$ with $D(z) \neq 0$. This implies that for $z_0 \in G$, any closed ball around z_0 only contains finitely many $z \in G$ with $D(z) \neq 0$. Making the radius of the balls small enough, it follows that there is a ball around z_0 that contains at most one of the $z \in G$ with $D(z) \neq 0$, so z_0 is not an accumulation point. □

Definition 5.16. Let $G \subset \mathbb{C}$ be a region and $f \in \mathcal{M}(G) \setminus \{0\}$ a meromorphic function. For $z \in G$, the **order** of f is

$$\operatorname{ord}_z(f) := \begin{cases} 0 & f \text{ holomorphic at } z \text{ with } f(z) \neq 0 \\ k, & f \text{ has a zero of order } k \text{ at } z \\ -k & f \text{ has a pole of order } k \text{ at } z. \end{cases}.$$

By Theorem 1.12 and the definition of meromorphic functions, the map $\operatorname{ord}(f)\colon G \to \mathbb{Z}$ is a divisor, called the **divisor of** f and is denoted by (f). A divisor D is called a **principal divisor**, if there exists $f \in \mathcal{M}(G) \setminus \{0\}$, such that $D = (f)$.
With pointwise addition, the set of divisors $\operatorname{Div}(G)$ becomes an abelian group (a subgroup of $\{G \to \mathbb{Z}\}$) and we also write a divisor $D\colon G \to \mathbb{Z}$ as a formal sum $\sum_{z\in G} D(z) \cdot z$.

Note that if G is compact, any divisor has finite support and thus $\operatorname{Div}(G)$ is the free abelian group generated by the points of G.
It is also worth mentioning that $\operatorname{Div}(G)$ is not a ring with pointwise multiplication, since the constant 1 function is not a divisor.

Example 5.17. For example, the principal divisor on G of $f \in \mathcal{M}(G) \setminus \{0\}$ is

$$(f) = \sum_{z\in G} \operatorname{ord}_z(f) \cdot z.$$

In particular, for a rational function $f(z) = \frac{p(z)}{q(z)}$ with $p(z) = \prod_{j=1}^{r}(z - z_j)^{m_j}$, $q(z) = \prod_{j=1}^{s}(z - \xi_j)^{n_j}$ (where $z_j, \xi_j \in G$) we have

$$(f) = \sum_{j=1}^{r} m_j \cdot z_j - \sum_{j=1}^{s} n_j \cdot \xi_j.$$

Definition 5.18. A divisor $D\colon G \to \mathbb{Z}$ is called positive, if $\operatorname{im}(D) \subset \mathbb{N}$.

Note that any divisor D can be written as the difference of two positive divisors: $D = D \cdot \mathbb{1}_{\{D \geq 0\}} - (-D \cdot \mathbb{1}_{\{D<0\}})$.

Lemma 5.19. The map

$$\operatorname{ord} : \mathcal{M}^*(G) \to \operatorname{Div}(G),\ f \mapsto (f)$$

is a group homomorphism. Its image is the set of principal divisors $\operatorname{HDiv}(G)$, which thus forms a subgroup of $\operatorname{Div}(G)$.
Furthermore, $f \in \mathcal{M}^*(G)$ is holomorphic on G if and only if $(f) \geq 0$ and invertible in $\mathcal{O}(G)$ if and only if $(f) = 0$.

In particular, the positive principal divisors are precisely those induced by holomorphic functions.

Definition 5.20. The quotient group $\operatorname{Div}(G)/\operatorname{HDiv}(G)$ is called the **divisor class group**.

Roughly, this group measures how much the divisors "differ" from being only principal divisors.
We can now restate Theorem 5.13 for divisors.

Theorem 5.21 (Weierstrass Factorization Theorem for Divisors).
On $\mathbb{C}$, the divisor class group is trivial; i.e. every divisor is a principal divisor.

The divisor class group is also trivial on the Riemann sphere, because the meromorphic functions on it are the rational functions (see Theorem 2.28).

6 The Riemann Mapping Theorem

Lec 10
2023-06-21

6.1 The Topology of Compact Convergence

Definition 6.1. Let X be a topological space. An **exhaustion by compact sets** of X is a nested sequence of compact sets

$$K_0 \subset K_1 \subset K_2 \subset \dots,$$

that covers X (i.e. $\bigcup_{i=0}^{\infty} K_i = X$) and such that each K_i is contained in the interior of K_{i+1}.

Intuitively, the second condition means that the minimal distance of the boundary of K_{i+1} from K_i is positive.

We will only apply this terminology to $\mathbb{C}$ (equipped with the usual topology).

Example 6.2. An exhaustion of $\mathbb{C}$ by compact sets is given by the closed balls $K_j \coloneqq \overline{B_j(0)}$. Similarly, an exhaustion of the open set $B_1(0) \subset \mathbb{C}$ is $K_j \coloneqq \overline{B_{1-\frac{1}{j}}(0)}$. Note that this second example can be obtained from the first one by applying a homeomorphism $\mathbb{C} \to B_1(0)$ that maps $B_j(0)$ to $B_{1-\frac{1}{j}}(0)$ for all $j \in \mathbb{N}_{>0}$.

Proposition 6.3. Every open subset $U \subset \mathbb{C}$ admits an exhaustion by compact sets.

Proof. A dense subset of U is $S \coloneqq U \cap (\mathbb{Q} + i\mathbb{Q})$. For every $q \in S$, we consider $s(q) \coloneqq \sup\left\{r \in [0,1] : \overline{B_r(q)} \subset U\right\} \in (0,1]$ and choose a strictly increasing nonzero sequence $s(q)_k \to s(q)$ converging to $s(q)$ from below. Because S is countable, we may write it as $S = \{q_j : j \in \mathbb{N}\}$. Then an exhaustion by compact sets of U is given by

$$K_k \coloneqq \bigcup_{j=0}^{k} \overline{B_{s(q_j)_k}(q_j)}.$$

As an alternative argument, one can note that $R \coloneqq \left\{\overline{B_r(q)} \subset U : r \in \mathbb{Q}_{>0}, q \in S\right\}$ is countable and choose an enumeration $R = \{R_j : j \in \mathbb{N}\}$. Then one defines recursively $K_1 \coloneqq R_1$ and $K_{k+1} \coloneqq \bigcup_{j=1}^{L} R_j$, where $L \in \mathbb{N}$ is chosen large enough, such that $K_k \subset \bigcup_{j=1}^{L} \operatorname{int}(R_j)$, which is always possible by compactness. ☐

We want to define a topology on the $\mathbb{C}$-algebra $\mathcal{O}(U)$ of holomorphic functions on U. For this, we first define one on the algebra of continuous functions $C(U,\mathbb{C})$, which contain $\mathcal{O}(U)$ as a subalgebra.

Definition 6.4. For $K \subset U$ compact, a seminorm on $C(U,\mathbb{C})$ is given by the supremum norm w.r.t. K; i.e.

$$p_K \colon C(U,\mathbb{C}) \to [0,\infty), \ f \mapsto \sup_{z\in K}|f(z)|.$$

Of course, this is not a norm, since any function $f \in C(U,\mathbb{C})$ that is zero on K satisfies $p_K(f) = 0$.
The smaller the chosen $K \subset U$ is, the "fewer" information p_K reveals about the continuous functions, since it only considers each function on K. For example, if $K = \{u\}$ is just a point, then $p_K(f) = |f(u)|$.

This also translates to the topologies $\mathcal{O}_K$ induced by each of the seminorms p_K. If $K \subset K'$ are two compact sets in U, then a set that is open w.r.t. p_K is also open w.r.t. $p_{K'}$; that is, $\mathcal{O}_K \subset \mathcal{O}_{K'}$. In other words, the topology $\mathcal{O}_{K'}$ is finer than $\mathcal{O}_K$, so in this sense, it yields more information.

Definition 6.5. We equip $C(U,\mathbb{C})$ with the **topology of compact convergence** that is generated by the basis

$$\bigcup_{K \subset U \text{ compact}} \mathcal{O}_K = \{V \subset C(U,\mathbb{C}) : \exists\, K \subset U \text{ compact} : V \text{ open w.r.t. } p_K\};$$

that is, $V \subset C(U,\mathbb{C})$ is open if and only if for every $f \in V$, there exists a compact subset $K \subset U$ and $\epsilon > 0$, such that $\{g \in C(U,\mathbb{C}) : p_K(f-g) < \epsilon\} \subset V$.
As a $\mathbb{C}$-subspace of $C(U,\mathbb{C})$, the $\mathbb{C}$-algebra of holomorphic functions $\mathcal{O}(U)$ inherits this topology.

Because we have $\mathcal{O}_K \subset \mathcal{O}_{K'}$ for $K \subset K'$, it follows that this topology can also be described as the topology generated by $\bigcup_{j=0}^{\infty} \mathcal{O}_{K_j}$, where (K_j) is an an arbitrary exhaustion by compact sets of U.
In the language of category theory, this can also be described as the limit of the functor

$$(\{K \subset U \text{ compact}\}, \subset) \to \mathcal{Top},\ K \mapsto (C(U,\mathbb{C}), \mathcal{O}_K),\ (K \subset K') \mapsto \mathrm{id}_{C(U,\mathbb{C})},$$

where we view the partially ordered set $(\{K \subset U \text{ compact}\}, \subset)$ as a category with a single morphism $K \to K'$ if and only if $K \subset K'$. Here $\mathcal{Top}$ denotes the category of topological spaces and continuous maps. By choosing an exhaustion (K_j) by compact sets of U, it can equivalently be described as the inverse limit

$$\ldots \xrightarrow{\mathrm{id}} (C(U,\mathbb{C}), \mathcal{O}_{K_2}) \xrightarrow{\mathrm{id}} (C(U,\mathbb{C}), \mathcal{O}_{K_1}) \xrightarrow{\mathrm{id}} (C(U,\mathbb{C}), \mathcal{O}_{K_0}).$$

For $f \in C(U,\mathbb{C})$, a neighborhood basis of f is given by the collection of open balls $B_\epsilon^K(f) \coloneqq \{g \in C(U,\mathbb{C}) : p_K(f-g) < \epsilon\}$ around f w.r.t. all of the p_K seminorms

$$\left\{B_\epsilon^K(f) \subset C(U,\mathbb{C}) : K \subset U \text{ compact}, \epsilon > 0\right\}.$$

In particular, the topological space is first countable, since a countable neighborhood basis can be obtained by choosing an exhaustion (K_n) by compact sets of U:

$$\left\{B_{\frac{1}{n}}^{K_n}(f) \subset C(U,\mathbb{C}) : n \in \mathbb{N}_{>0}\right\}.$$

In fact, $C(U,\mathbb{C})$ is even metrizable. In order to see this, we use general results from category theory and topology. One first checks that $C(U,\mathbb{C})$ with the topology of compact convergence is the limit of the diagram (in $\mathcal{Top}$)

$$\ldots \longrightarrow (C(K_2,\mathbb{C}), \mathcal{O}_{K_2}) \longrightarrow (C(K_1,\mathbb{C}), \mathcal{O}_{K_1}) \longrightarrow (C(K_0,\mathbb{C}), \mathcal{O}_{K_0}),$$

where $\mathcal{O}_{K_i}$ is just the topology induced by the supremums norm on K_i and every continuous map is a restriction. By the universal property of the product $\prod_{j=0}^{\infty}(C(K_j,\mathbb{C}), \mathcal{O}_{K_j})$, there exists a canonical map $C(U,\mathbb{C}) \to \prod_{j=0}^{\infty}(C(K_j,\mathbb{C}), \mathcal{O}_{K_j})$, mapping $f \in C(U,\mathbb{C})$ to the sequence with j-th component $f|_{K_j}$ and this is seen to be a topological embedding.

Lemma 6.6. Let $(X_n)_{n\in\mathbb{N}_{>0}}$ be a sequence of metrizable topological spaces. Denoting a metric of X_n by d_n, their product $\prod_{n=1}^{\infty} X_n$ is metrizable with metric

$$d(x,y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n,y_n)}{1+d_n(x_n,y_n)}.$$

Because the above embedding allows us to view $C(U,\mathbb{C})$ with the topology of compact convergence as a subspace of $\prod_{j=0}^{\infty}(C(K_j,\mathbb{C}),\mathcal{O}_{K_j})$, we obtain the following result.

Theorem 6.7. The set of continuous functions $C(U,\mathbb{C})$ (and thus $\mathcal{O}(U)$) equipped with the topology of compact convergence is metrizable. Choosing an exhaustion (K_n) by compact sets of U, a metric inducing the topology is given by

$$d(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_{K_n}(f-g)}{1+p_{K_n}(f-g)}.$$

Furthermore, it can be shown that the topology of compact convergence is just the compact-open topology and that $\mathcal{O}(U)$ becomes a locally convex topological vector space.

6.2 Compact Convergence

We now investigate convergence in this metrizable topological space.

Definition 6.8. Let (f_j) be a sequence of functions in $C(U,\mathbb{C})$ (or $\mathcal{O}(U)$) that converges to $f \in C(U,\mathbb{C})$ w.r.t. the topology of compact convergence; that is,

$$p_K(f_j - f) = \sup_{z\in K}|f_j(z) - f(z)| \to 0 \qquad \forall\ K \subset U \text{ compact}.$$

We say that f_j is **compactly convergent** to f.

Clearly any compactly convergent sequence is also pointwise convergent.
It should also be mentioned that this notion of convergence is equivalent to *local uniform convergence*, where one asks that for each point $z_0 \in U$, there exists a neighborhood of it on which the (f_j) converge uniformly. This is true because any compact set is covered by finitely many of such neighborhoods and on the other hand any such neighborhood contains a compact ball.

As a consequence of the next theorem, the limit of a compactly convergent sequence of holomorphic functions is automatically holomorphic. See [Bor16, Thm 3.3.2] for a proof. Another immediate consequence is that taking derivatives $\mathcal{O}(U) \to \mathcal{O}(U),\ f \mapsto f'$ constitutes a continuous function.

Theorem 6.9 (**Weierstrass Convergence Theorem**).
If a sequence $(f_k) \in \mathcal{O}(U)$ compactly converges to a function $f\colon U \to \mathbb{C}$, then $f \in \mathcal{O}(U)$. Furthermore, all derivatives $(f_k^{(n)})$ compactly converge to $(f^{(n)})$.

Our next goal is to characterize the compact sets in $\mathcal{O}(U)$. For this, we use the following lemma from category theory and topology.

Lemma 6.10. Let $\lim(F)$ denote the limit of a functor $F\colon \mathcal{C} \to \mathcal{Top}$, which by definition comes equipped with continuous maps $\pi_C\colon \lim(F) \to F(C)$ for $C \in \mathcal{C}$. Then a net (in particular a sequence) $(x_n) \in \lim(F)$ converges to $x \in \lim(F)$ if and only if for all $C \in \mathcal{C}$, the nets $(\pi_C(x_n))$ converge to $(\pi_C(x))$ in $F(C)$.

Proof. Any limit in $\mathcal{T}op$ carries the initial topology. This can be shown using the characterstic property of the initial topology (see here). Using the definition of the initial topology, it is straightforward to verify that a net in such a topological space converges if and only if its images converge. □

We spell out the special case of the previous general lemma that we will make use of. It can also be proven directly, without any appeals to category theory.

Lemma 6.11. Let $U \subset \mathbb{C}$ be open and $(f_k) \in \mathcal{O}(U)$ a sequence of holomorphic functions. (f_k) is compactly convergent if and only if for every compact subset $K \subset U$, the restrictions $(f_k|_K)$ converge in $(C(K, \mathbb{C}), \|\cdot\|_\infty)$.
In that case, denoting the limit of $(f_k|_K)$ by $g_K \in C(K, \mathbb{C})$ and the limit of f_k in $\mathcal{O}(U)$ by g, we have $g|_K = g_K$ for all $K \subset U$ compact.

As a small application of this lemma, we show that $\mathcal{O}(U)$ is complete.

Theorem 6.12. With the metric from Theorem 6.7, $\mathcal{O}(U)$ is complete.

Proof. For an exhaustion (K_n) of U, we have the estimate

$$\frac{1}{2^{n+1}} p_{K_n}(f-g) \leq \frac{1}{2^n} \frac{p_{K_n}(f-g)}{1+p_{K_n}(f-g)} \leq d(f,g) \qquad \forall\ f,g, \in \mathcal{O}(U).$$

Therefore, the restriction maps $\pi_n \colon (\mathcal{O}(U), d) \to (C(K_n, \mathbb{C}), \|\cdot\|_\infty)$ are Lipschitz continuous. It follows that any Cauchy sequence $(f_k) \in \mathcal{O}(U)$ translates to a Cauchy sequence $(\pi_n(f_k))_k$ for all $n \in \mathbb{N}$. Because any compact set $K \subset U$ is contained in one of the K_n, the sequence $(\pi_K(f_k))_k$ is a Cauchy sequence in $(C(K, \mathbb{C}), \|\cdot\|)$. Since $(C(K, \mathbb{C}), \|\cdot\|_\infty)$ is a Banach space, those sequences converge and thus Lemma 6.11 implies that (f_k) must be convergent as well. □

Definition 6.13. A subset $A \subset \mathcal{O}(U)$ is called **bounded**, if for all compact sets $K \subset U$, the set $p_K(A) = \{\sup_{z\in K}|f(z)| : f \in A\}$ is bounded in $\mathbb{R}$; that is, if for any compact set $K \subset U$, there exists $C \in \mathbb{R}_{>0}$ such that $|f(z)| < C$ for all $f \in A$ and $z \in K$.

It should be noted that this is not the notion of bounded that one would obtain from the above metric, because that metric satisfies $d(f,g) \leq 1$ for all $f, g \in \mathcal{O}(U)$ and thus any set is bounded w.r.t. it.

With this terminology, we can generalize a familiar fact from $\mathbb{R}^n$, namely that the compact sets are precisely the closed and bounded ones. The proof of the corresponding theorem requires yet another lemma.

Lemma 6.14. Let $U \subset \mathbb{C}$ be open, $K \subset U$ compact and $A \subset \mathcal{O}(U)$ be bounded. Then A is uniformly locally Lipschitz continuous on K in the sense that there exists $c > 0$, $r > 0$, such that $|f(z) - f(w)| \leq c \cdot |z - w|$ for all $z, w \in K$ with $w \in B_r(z)$ and $f \in A$.

Proof. As a compact set, K has a positive "minimal distance" $d(K, \partial U) > 0$ from the boundary ∂U of U, which is closed and disjoint and that we may assume to be nonempty. Thus there exists a compact set $K^* \subset U$ and $r > 0$, such that $B_{2r}(w) \subset K^*$ for all $w \in K$. Let $z, w \in K$ be arbitrary with $|z - w| < r$. Then any $v := w + t(z - w)$ for $t \in [0, 1]$ satisfies $v \in K$ and $B_r(v) \subset K^*$. Cauchy's inequality (see [Bor16, Thm 3.2.1]) shows that $z \in K$ admits the estimate

$$|f'(z)| \leq \frac{1}{r}\|f\|_{\partial B_r(z)} \leq \frac{1}{r} p_{K^*}(f) \leq \frac{1}{r} \sup_{g\in A} p_{K^*}(g).$$

Since

$$f(z) - f(w) = \int_0^1 f'(w + t(z-w))dt \cdot (z-w),$$

we conclude that

$$|f(z) - f(w)| \leq \frac{1}{r} \sup_{g \in A} p_{K^*}(g) \cdot |z-w|. \qquad \square$$

Lec 11
2023-06-28

Theorem 6.15 (**Montel**)**.** $A \subset \mathcal{O}(U)$ is compact if and only if it is closed and bounded.

Proof. Suppose that $A \subset \mathcal{O}(U)$ is compact. Because $p_K \colon \mathcal{O}(U) \to \mathbb{R}$ is continuous, the image $p_K(A) \subset \mathbb{R}$ is compact and thus bounded. Furthermore, A is closed as a compact subset of a Hausdorff space.

On the other hand, let $A \subset \mathcal{O}(U)$ be closed and bounded. Since the topology is metrizable, it suffices to show that A is sequentially compact; that is, that any sequence $(f_j) \in A$ has a convergent subsequence. To prove this, we use a *diagonal argument*.

Let $(q_j) \subset U$ be a sequence whose image in U is dense (e.g. $U \cap (\mathbb{Q} + i\mathbb{Q})$). We may choose a subsequence $(f_{1,j})$ of f_j, such that the sequence $(f_{1,j}(q_1))$ converges in $\mathbb{C}$. This subsequence admits another subsequence $(f_{2,j})$, such that $(f_{2,j}(q_2))$ converges in $\mathbb{C}$. Because it is a subsequence of the first one, $(f_{2,j}(q_1))$ converges as well. Iterating this, we obtain a sequence of subsequences $f_{k,j}$ with the property that $(f_{k,j}(q_l))_j$ converges for $l \leq k$. Thus the sequence $(g_k) \coloneqq (f_{k,k})$ is a subsequence of (f_j), such that $\lim_{k\to\infty} g_k(q_j) \in \mathbb{C}$ exists for every $j \in \mathbb{N}_{>0}$.

Let $K \subset U$ be a compact subset. By Lemma 6.14, we can choose $r > 0$ and $c > 0$, such that

$$|f(z) - f(w)| \leq c \cdot |z-w| \qquad \forall\ z, w \in K, w \in B_r(z), f \in A.$$

For given $\epsilon > 0$, we set $C \coloneqq \min\{r, \frac{\epsilon}{3c}\} > 0$. Because K is compact and (q_j) is dense, the cover $\{B_C(q_j) : j \in \mathbb{N}\}$ of K admits a finite subcover; that is, there exists $L \in \mathbb{N}$, such that for all $z \in K$ there is $k \leq L$ with $|z - q_k| < C$. By the above, there exists $N \in \mathbb{N}$, such that

$$|g_n(q_k) - g_m(q_k)| < \frac{\epsilon}{3} \qquad \forall\ m, n \geq N, k \leq L.$$

Because any $z \in K$ satisfies $|z - q_k| < C$ for some $k \leq L$, it follows for $m, n \geq N$ that

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(q_k)| + |g_n(q_k) - g_m(q_k)| + |g_m(q_k) - g_m(z)| < \epsilon.$$

We conclude that $p_K(g_n - g_m) < \epsilon$ for all $m, n \geq N$, so (g_n) is a Cauchy sequence in the Banach space $(C(K, \mathbb{C}), \|\cdot\|_\infty)$. By Theorem 6.9, the sequence also converges in $\mathcal{O}(U)$ and the limit lies in A as it is closed. $\square$

6.3 Univalent Functions

Definition 6.16. Let $U \subset \mathbb{C}$ be open. A **conformal map** $f \colon U \to \mathbb{C}$ is a holomorphic function $f \in \mathcal{O}(U)$, such that $f' \colon U \to \mathbb{C}$ has no zeros.

As a consequence of Theorem 1.6, a conformal map locally preserve angles and orientations at every point.

Definition 6.17. For $U \subset \mathbb{C}$ open, an injective holomorphic function $f \colon U \to \mathbb{C}$ is called **univalent**.

By Theorem 1.15, a univalent function is a topological embedding. In fact, even more is true.

Theorem 6.18 ([Bor16, Thm 7.4.1])**.** Let $G \subset \mathbb{C}$ be a region and $f \in \mathcal{O}(G)$ univalent. Then f is biholomorphic (onto its image) and conformal.

We now show that compact convergence preserves the injectivity of the functions.

Theorem 6.19 (**Hurwitz**)**.**
Let $G \subset \mathbb{C}$ be a region and (f_j) a sequence of univalent functions on G. If (f_j) converges compactly to a nonconstant $f \in \mathcal{O}(G)$, then f is univalent.

Proof. By contraposition, assume that there exist $u, v \in G$ with $u \neq v$ and $f(u) = f(v)$. By replacing f with $f - f(u)$, we may assume that u and v are distinct zeros of f. According to Theorem 1.12, we can choose $r > 0$ be small enough, such that $K := \overline{B_r(u)} \subset G$ and that f has no other zero (except u) on this set. Then the residuum of the logarithmic derivative $\frac{f'}{f}$

$$m := \frac{1}{2\pi i} \int_{\partial B_r(u)} \frac{f'(\xi)}{f(\xi)} d\xi \in \mathbb{N}_{>0}$$

equals the order of the zero at u (see [Bor16, Thm 5.5.1]). Because (f_j) is compactly convergent to f, the same is true for their derivatives by Theorem 6.9, so the sequence (f_j') converges uniformly to f' on K. Therefore, we can find $N \in \mathbb{N}$, such that for $j \geq N$, f_j has no zeros on ∂K and

$$\frac{1}{2\pi i} \int_{\partial B_r(u)} \frac{f_j'(\xi)}{f_j(\xi)} d\xi = m.$$

It follows (by the same theorem as above) that for all large enough $j \in \mathbb{N}$, the function f_j has at least one zero in the interior of $B_r(u)$.
With the same argument, we find a neighborhood $V \subset G \setminus K$ of v such that for all large enough $j \in \mathbb{N}$, f_j has a zero in V. This shows that only finitely many of the f_j are univalent. □

Lemma 6.20 (**Lemma of Schwarz**)**.**
Let $f\colon B_1(0) \to B_1(0)$ be holomorphic with $f(0) = 0$. Then $|f(z)| \leq |z|$ for $z \in B_1(0)$ and $|f'(0)| \leq 1$.
Furthermore, if $|f(z_0)| = z_0$ for some $z_0 \in B_1(0) \setminus \{0\}$ or $|f'(0)| = 1$, then f is just given by scalar multiplication with a scalar $\lambda \in S^1(0)$ (i.e. $f(z) = \lambda \cdot z$).

Proof. Consider the holomorphic function

$$g\colon B_1(0) \setminus \{0\} \to B_1(0),\ z \mapsto \frac{f(z)}{z}$$

and note that $\lim_{z\to 0} g(z) = f'(0)$. Setting $g(0) := f'(0)$, the *Riemann removable singularity theorem* (see [Bor16, Thm 3.5.2]) implies that $g \in \mathcal{O}(B_1(0))$.
For $r \in (0, 1)$, the *maximum modulus principle* (see [Bor16, Cor 3.4.3]) yields $\zeta \in \partial B_r(0)$, such that

$$\left|\frac{f(z)}{z}\right| = |g(z)| \leq |g(\zeta)| = \left|\frac{f(\zeta)}{\zeta}\right| \leq \frac{1}{r} \qquad \forall\ z \in B_r(0) \setminus \{0\}.$$

Taking the limit $r \to 1$ from below yields $\frac{|f(z)|}{|z|} \leq 1$ for $z \in B_1(0)$.
Moreover, if $z_0 \in B_1(0)\setminus\{0\}$ satisfies $|f(z_0)| = |z_0|$, it follows that $|g(z_0)| = 1$ and because $|g| \leq 1$ on $B_1(0)$, the maximum modulus principle implies that g is a constant $\lambda \in S^1$. The same argument applies if instead $|f'(0)| = 1$, since then $|g(0)| = 1$. □

Intuitively, the lemma tells us that holomorphic functions preserving the unit ball cannot grow fast; they cannot enlarge the modulus of their input.

We require one more lemma before we can prove the Riemann mapping theorem.

Lemma 6.21. Let $G \subset \mathbb{C}$ be a simply connected region $f \in \mathcal{O}^*(G)$ and $k \in \mathbb{N}_{>0}$. Then there exists $g \in \mathcal{O}(G)$, such that $f = g^k$ (i.e. $f(z) = g(z)^k$ for all $z \in G$).
g is called **k-th root of f on G.**

Proof. By Lemma 5.11, there exists $h \in \mathcal{O}(G)$ such that $f = \exp \circ h$ and thus $g(z) \coloneqq \exp\left(\frac{h(z)}{k}\right) \in \mathcal{O}(G)$ is the desired function. □

6.4 The Riemann Mapping Theorem

Theorem 6.22 (Riemann Mapping Theorem).
Every simply connected region $G \subsetneq \mathbb{C}$ is biholomorphically equivalent to $B_1(0)$; i.e. there exists a biholomorphic function $f\colon G \to B_1(0)$.

Proof. We differentiate three cases for the simply connected region G.
Case 1: $G \subset B_1(0)$.
Since any translation $z \mapsto z + c$ is a biholomorphic map, we may assume that $0 \in G$. Consider the nonempty subset

$$A \coloneqq \{f \in \mathcal{O}(G) : f(G) \subset B_1(0), f \text{ univalent}, f(0) = 0, |f'(0)| \geq 1\} \subset \mathcal{O}(G).$$

We claim that it is compact. By Theorem 6.15, since A is clearly bounded, it suffices to show that A is closed. Let $f_j \to f$ be a convergent sequence in A. To see that $f \in A$, we first observe using Theorem 6.9 that $f \in \mathcal{O}(G)$, $f(0) = 0$ and $|f'(0)| = \lim_{j\to\infty}\left|f_j'(0)\right| \geq 1$. Furthermore, because f is not constant, $f(G) \subset \overline{B_1(0)}$ is open (in $\mathbb{C}$) by Theorem 1.15, so $f(G) \subset B_1(0)$. Using Theorem 6.19, it follows that f is univalent and thus $f \in A$. Because the map $f \mapsto |f'(0)|$ is continuous (by Theorem 6.9) the compactness of A yields the existence of $g \in A$ with

$$|g'(0)| = \sup_{f\in A}|f'(0)| \geq 1.$$

We claim that $g(G) = B_1(0)$, so that $g\colon G \to B_1(0)$ is biholomorphic by Theorem 6.18. Assume this is not the case; i.e. there exists $a \in B_1(0) \setminus g(G)$. Because $a \neq 0$, we can choose $b \in B_1(0)$ with $b^2 = a$. For $c \in B_1(0)$, we consider the function

$$\phi_c\colon B_1(0) \to B_1(0),\ z \mapsto \frac{z - c}{\bar{c}z - 1}$$

and note that it is biholomorphic, self-inverse and satisfies $\phi_c(c) = 0$.
We construct the functions Lec 12 2023-07-06

$$p \coloneqq \phi_a \circ {\phi_b}^2 \in \mathcal{O}(B_1(0)), \qquad q \coloneqq \phi_a \circ g \in \mathcal{O}(G),$$

where ${\phi_b}^2$ denotes the function $z \mapsto \phi_b(z)^2$. Observe that $p(0) = 0$, $p(B_1(0)) \subset B_1(0)$ and that p is not injective (as e.g. $p({\phi_b}^{-1}(\frac{1}{2})) = p({\phi_b}^{-1}(-\frac{1}{2})))$, so Lemma 6.20 implies

$|p'(0)| < 1$. Because G is a simply connected region and $0 \notin q(G)$, Lemma 6.21 yields $h \in \mathcal{O}(G)$ with $h^2 = q$. Since $q(0) = a$, it follows that $h(0) \in \{b, -b\}$, so by potentially replacing h with $-h$, we may assume that $h(0) = b$. Additionally, h is univalent as the same holds true for q.
Using these observations, it follows that the function $f := \phi_b \circ h \in \mathcal{O}(G)$ is well-defined, univalent and satisfies $f(G) \subset B_1(0)$ and $f(0) = 0$. Because

$$(p \circ f)(z) = \big(\phi_a \circ \phi_b^2 \circ \phi_b \circ h\big)(z) = \phi_a\big(h(z)^2\big) = \phi_a(q(z)) = g(z)$$

we have $p'(0) \cdot f'(0) = g'(0)$, so that

$$|f'(0)| = \left|\frac{g'(0)}{p'(0)}\right| > |g'(0)| \geq 1,$$

showing that $f \in A$ and contradicting the definition of g as a supremum. This establishes the theorem in the special case that $G \subset B_1(0)$.
Case 2: G is not dense in $\mathbb{C}$.
By assumption, there exists $a \in \mathbb{C}$ and $\epsilon > 0$, such that $|z - a| > 2\epsilon$ for all $z \in G$. Let $b \in G$ be arbitrary. The function $f(z) := \frac{\epsilon}{z-a} \in \mathcal{O}(G)$ is univalent and satisfies $f(G) \subset B_{\frac{1}{2}}(0)$, so by Theorem 6.18, $g(z) := f(z) - f(b) \in \mathcal{O}(G)$ constitutes a biholomorphic map onto its image $g(G) \subset B_1(0)$. By the first case, its image is biholomorphic to $B_1(0)$ and thus the assertion follows.
Case 3: $G \subset \mathbb{C}$ arbitrary simply connected region.
By translating $z \mapsto z + c$, we may assume that $0 \notin G$. Because G is simply connected, Lemma 6.21 yields the existence of $f \in \mathcal{O}(G)$ with $f^2 = \mathrm{id}_G$. Because f is univalent, Theorem 6.18 implies that f is a biholomorphic map onto its image $H := f(G)$. The set $H \cap (-H) = \{c \in G : c \in H, -c \in H\}$ must be empty; for if not, there exist $c, u, v \in G$, such that $c = f(u) = -f(v)$ and thus $u = f(u)^2 = (-f(v))^2 = v$, so that $c = 0$.
In particular, any ball $B_\epsilon(z) \subset H$ with $B_\epsilon(z) \subset \mathbb{R}_+ + i\mathbb{R}$ gives rise to the ball $B_\epsilon(-z)$ with $B_\epsilon(-z) \cap H = \emptyset$. Consequently, H is not dense in $\mathbb{C}$ and the second case yields the claim. □

Note that by Theorem 1.14, $\mathbb{C}$ and $B_1(0)$ are not biholomorphically equivalent.

7 Sheaf Cohomology in the Complex Plane

7.1 Sheaves

Sheaves are a tool from category theory that essentially describes local properties. They are of key importance to *algebraic geometry* and also appear in complex analysis, as we will see.

Definition 7.1. Let X be a topological space. Its open sets $\mathcal{O}$ are partially ordered by inclusion and thus form a category $(\mathcal{O}, \subset)$.
A **presheaf** F of X in a category $\mathcal{C}$ is a functor $F\colon (\mathcal{O}, \subset)^{\mathrm{op}} \to \mathcal{C}$. We restrict ourselves to the case that $\mathcal{C} = \mathcal{Ab}$ is the category of abelian groups. Explicitly, it consists of the following data:

(a) For every open set $U \subset X$, we have an abelian group $F(U)$.

(b) For every inclusion $V \subset U$ of open sets V, U in X, there is a homomorphism of abelian groups

$$\varphi_{U,V}\colon F(U) \to F(V),$$

such that

(1) $\varphi_{U,U}\colon F(U) \to F(U)$ is the identity $\mathrm{id}_{F(U)}$.

(2) For the inclusion $W \subset V \subset U$ of open sets, it holds $\varphi_{U,W} = \varphi_{V,W} \circ \varphi_{U,V}$.

The elements of $F(U)$ are called *sections* of F over U and the group homomorphisms $\varphi_{U,V}$ are called *restriction homomorphisms*.
It is common to write $s|_V := \varphi_{U,V}(s)$ for $s \in F(U)$ when U is clear from the context.

Roughly, this construction allows attaching groups to the open sets of a topological space in such a way that passing to a smaller open set corresponds to passing to a different group.

Definition 7.2. A presheaf F on a topological space X is called a **sheaf** if the following two conditions are satisfied:

(a) *Separateness*: If $U \subset X$ is open, $\{V_i\}_{i\in I}$ is an open cover of U and $s \in F(U)$, such that $s|_{V_i} = 0$ for all $i \in I$, then $s = 0$.

(b) *Glueing of sections*: If $U \subset X$ is open, $\{V_i\}_{i\in I}$ is an open cover of U and there are $s_i \in F(V_i)$ such that for all $i, j \in I$, we have $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$, then there exists an element $s \in F(U)$ with $s|_{V_i} = s_i$ for all $i \in I$.

A presheaf that satisfies the first condition is called *separated*.

The s from the second condition is unique by the first condition and a sheaf satisfies $F(\emptyset) = 0$. Intuitively, a presheaf is a sheaf if it allows passing from "local" to "global" information.

Example 7.3. Let X be a topological space and A an abelian group.

(a) The most important class of sheaves are *function sheaves*. These are sheaves assigning to $U \subset X$ a certain subset of functions from U to some fixed abelian group G and whose restriction morphisms are given by function restriction.
For example, the functor $F(U) \coloneqq \{\text{continuous functions } U \to \mathbb{C}\}$ (or to $\mathbb{R}$) with restricting maps as its action on morphisms defines the *sheaf of continuous functions*. If X is a smooth manifold, one can analogously define the sheaf of differentiable functions.
Such functors are always presheaves; they are sheaves if and only if the condition imposed is of "local" nature; i.e. if it can be checked locally on an arbitrary small neighborhood of a given point. For instance, the functor $F(U) \coloneqq \{\text{constant functions } U \to \mathbb{R}\}$ is generally not a sheaf, since "being constant" is a global property and not a local one. Here being a local property means that the property can be checked on an arbitrary small neighborhood.

(b) The functor

$$F(U) \coloneqq \begin{cases} A & U \neq \emptyset \\ 0 & \text{otherwise} \end{cases},$$

acting via the identity $\varphi_{U,V} = \mathrm{id}_A$ for $\emptyset \neq V \subset U$ is a presheaf on X but generally not a sheaf. This is the "constant function" presheaf from the previous example.

(c) Of central importance in complex analysis is the *sheaf of holomorphic functions* $\mathcal{O}$ on an open subset $X \subset \mathbb{C}$, where $\mathcal{O}(U)$ (for $U \subset X$ open) denotes the group of holomorphic functions $U \to \mathbb{C}$ and the restriction morphisms are just function restriction.

(d) Of similar importance is the *sheaf of meromorphic functions* $\mathcal{M}$ on an open set $X \subset \mathbb{C}$, assigning to an open set $U \subset X$ the set of meromorphic functions $\mathcal{M}(U)$ on that set. Note that this is not a function sheaf in the above sense (at least not for $G = \mathbb{C}$).

(e) The *constant sheaf* $\mathrm{const}(A)$ associated to A is the function sheaf

$$G(U) \coloneqq \{\text{locally constant functions } U \to A\}.$$

If X is locally connected (i.e. all connected components of all open subsets $U \subset X$ are open), then one can show that this functor is isomorphic to the functor assigning to $U \subset X$ the group $\prod_{i \in I} A$ for the decomposition $U = \coprod_{i \in I} U_i$ of U into connected components U_i.

Lec 13 2023-07-12

We now introduce the notion of *stalks* of a presheaf, which capture the local nature of presheaves (and in particular sheaves). We first give the categorical definition, but for our purposes it suffices to understand the explicit construction (6).

Definition 7.4. Let X be a topological space, F a presheaf on X and $x \in X$ a point. The *stalk* F_x of F at x is the direct limit

$$F_x \coloneqq \varinjlim_{\substack{U \subset X \text{ open} \\ x \in U}} F(U).$$

By definition, it comes equipped with a canonical group homomorphism $F(U) \to F_x$ for every open neighborhood $U \subset X$ of x. The image of a section $s \in F(U)$ is denoted by s or by s_x. The elements of the stalk are called *germs*.

For a directed set $(I, \leq)$ and a functor $F\colon (I, \leq) \to \mathcal{C}$, a *cocone* is an object $B \in \mathcal{C}$ together with morphisms $F(i) \to B$ for $i \in I$, such that the diagram

$$\begin{array}{ccc} F(i) & \xrightarrow{F(i\leq j)} & F(j) \\ \downarrow & \swarrow & \\ B & & \end{array}$$

commutes for all $i, j \in I$ with $i \leq j$.
The *direct limit* of F (if existent) is a cocone (i.e. an object $\operatorname{colim}(F)$ with morphisms $F(i) \to \operatorname{colim}(F)$ for $i \in I$), satisfying the following universal property: For any cocone (i.e. an object $C \in \mathcal{C}$ with morphisms $f_i\colon F(i) \to C$ for $i \in I$), there exists a unique morphism $\operatorname{colim}(F) \to C$, such that the diagram

$$\begin{array}{ccc} F(i) & \longrightarrow & \operatorname{colim}(F) \\ & \searrow^{f_i} & \downarrow \exists ! \\ & & C \end{array}$$

commutes for all $i \in I$.
In algebraic categories such as $\mathcal{Ab}$, direct limits always exist. Explicitly, the underlying set is the "quotiented" disjoint union $\left(\coprod_{i\in I} F(i)\right)/\sim$, where $\sim$ is the equivalence relation such that for $x_i \in F(i)$, $x_j \in F(j)$:

$$x_i \sim x_j \ :\Longleftrightarrow\ \exists\ k \in I : k \geq i,\ k \geq j,\ F(i \leq k)(x_i) = F(j \leq k)(x_j).$$

This becomes an abelian group by defining

$$x_i + x_j := F(i \leq k)(x_i) + F(j \leq k)(x_j) \quad \text{ for } k \geq i, k \geq j.$$

The canonical group homomorphism $F(i) \to \operatorname{colim}(F)$ is induced by the inclusion $F(i) \hookrightarrow \coprod_{i\in I} F(i)$.

Applying this to the case of stalks, where I is the set of open neighborhoods of x ordered by inclusion, it follows that

$$F_x = \Bigg(\coprod_{\substack{U \subset X \text{open} \\ x \in U}} F(U) \Bigg) \Big/ \sim \tag{6}$$

with

$$(U, \sigma) \sim (V, \tau) \ :\Longleftrightarrow\ \exists\ W \subset U \cap V,\ W \text{ open},\ x \in W,\ \sigma|_W = \tau|_W .$$

In particular, germs can be represented as pairs (U, σ) with $U \subset X$ an open neighborhood of x and $\sigma \in F(U)$ a section. Two pairs represent the same equivalence class if and only if there is a smaller neighborhood $W \subset U \cap V$ such that σ and τ become equal in the corresponding group (i.e. $\sigma|_W = \tau|_W$).
Intuitively, if $s_x \in F_x$ has a certain property this just means that there exists an open neighborhood U of x and section $s \in F(U)$ having that same property.

Example 7.5.

(a) Consider a function presheaf F for a fixed abelian group G. The stalk of $x \in X$ consists of the functions agreeing in a neighborhood of x (with pointwise addition). There exists an *evaluation homomorphism* $F_x \to G$, $(s, U) \to s(x)$, which can be dedcued from the universal property of the stalk (a colimit). In this sense, the stalk at x of a function presheaf represents the "local behavior" of those functions at x.

(b) In particular, for the sheaf of holomorphic functions $\mathcal{O}$ on an open set $X \subset \mathbb{C}$, the stalk $\mathcal{O}_x$ is the abelian group consisting of all convergent Taylor series $\sum_{j=0}^{\infty} a_j(z-x)^j$ at x and the evaluation homomorphism is $F_x \to \mathbb{C}$, $\sum_{j=0}^{\infty} a_j(z-x)^j \mapsto a_0$.

(c) The stalk $\mathcal{M}_x$ of the sheaf of meromorphic functions $\mathcal{M}$ on an open set $X \subset \mathbb{C}$ is the abelian group of convergent Laurent series $\sum_{j=-N}^{\infty} a_j(z-x)^j$ at x.

7.2 Riemann Surfaces

Generalizing the Riemann sphere from Section 2, we introduce *Riemann surfaces*. The differentiable structure of a real manifold is easily translated to the complex case, by replacing "diffeomorphism" with "biholomorphic map". However, the definition might look quite artificial if one is not familiar with smooth manifolds.

Definition 7.6. A **topological complex manifold** X is a second-countable Hausdorff space such that every point admits a neighborhood U and a homeomorphism $\phi\colon U \to V$ with $U \subset X$ open and $V \subset \mathbb{C}^n$ open.
Such a homeomorphism is called a **(complex) chart** and is often denoted by (ϕ, U).
Two complex charts (ϕ, U) (ψ, V) are called **compatible** (or **biholomorphically equivalent**), if the **transition function**

$$\psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V)$$

is biholomorphic.

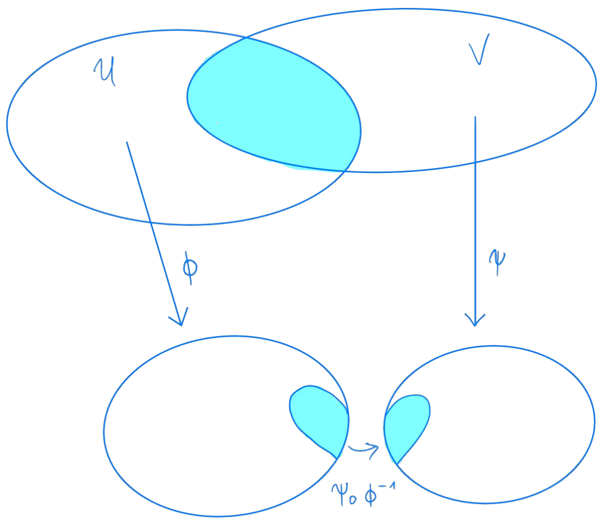

Figure 7: The transition functions of a manifold.

A **complex atlas** is a set of transition functions $\{(\phi_i, U_i)\colon i \in I\}$, such that any two are compatible and such that they cover X in the sense that $X = \bigcup_{i \in I} U_i$.
There is an equivalence relation $\sim$ on the set of all complex atlases on X, where

$$A \sim B \;:\Longleftrightarrow\; \forall\, (\phi_i, U_i) \in A, (\psi_i, V_i) \in B : \; (\phi_i, U_i) \text{ and } (\psi_i, V_i) \text{ are compatible.}$$

An equivalence class of complex atlases is called a **complex structure on X**.
An n-**dimensional complex manifold** X is a topological complex manifold X together with a complex structure on X.
A **Riemann surface** is a 1-dimensional connected complex manifold.

Just like a smooth structure on a smooth (real) manifold allows us to define when functions on that manifold are differentiable, the complex structure of a complex manifold allows us to define when a function on that manifold is holomorphic. We will give the corresponding definition after highlighting some examples.

Example 7.7.

(a) Any region $G \subset \mathbb{C}$ is a Riemann surface. Here the atlas consisting only of the inclusion $G \hookrightarrow \mathbb{C}$ gives a complex structure.
In particular, $\mathbb{C}$ itself is a Riemann surface.

(b) The Riemann sphere $\hat{\mathbb{C}}$ is a compact Riemann surface with an atlas consisting of the two charts

$$\mathrm{id}_{\mathbb{C}} \colon \mathbb{C} \to \mathbb{C}, \qquad J \colon \hat{\mathbb{C}} \setminus \{0\} \to \mathbb{C},$$

where J is the inversion map from Definition 2.9.

(c) The torus $S^1 \times S^1$ can be endowed with the structure of a Riemann surface.

Definition 7.8. For a Riemann surface X and $Y \subset X$ an open subset, a function $f \colon Y \to \mathbb{C}$ is called **holomorphic (meromorphic)**, if the composition $f \circ \phi^{-1} \colon \phi(U \cap Y) \to \mathbb{C}$ is holomorphic (meromorphic) for all charts (ϕ, U) in a complex atlas (of the complex structure) of X.
Extending the notation from Definition 1.8 and Definition 1.17, the set of all holomorphic functions $Y \to \mathbb{C}$ is denoted by $\mathcal{O}(Y)$ and the set of all meromorphic functions $Y \to \mathbb{C}$ by $\mathcal{M}(Y)$.

Note that this definition agrees with the usual definition of holomorhpic (and meromorphic) functions when we view a region $G \subset \mathbb{C}$ as a Riemann surface, since $f \circ \phi^{-1} = f$ for the inclusion $G \hookrightarrow \mathbb{C}$. Similarly, the definition agrees with Definition 2.12 if we consider the Riemann sphere as a Riemann surface (as above).
It is straightforward to see that Example 7.3 can be extended to define the sheaf of holomorphic $\mathcal{O}$ and the sheaf of meromorphic $\mathcal{M}$ functions for open subsets of any Riemann surface (instead of just $\mathbb{C}$).

7.3 Čech Cohomology

Lec 14 2023-07-19

We can finally introduce *Čech Cohomology*. For more details, see [For81, Ch. 12].

Definition 7.9. Let X be a topological space with an open cover $(U_i)_{i \in I}$ and F a presheaf on X. For $q \in \mathbb{N}$, the product abelian group

$$C^q(U_i, F) := \prod_{i \in I^{q+1}} F\left(\bigcap_{j=0}^{q} U_{i_j}\right)$$

is called the **abelian group of q-cochains**.
This means that a q-cochain is a collection (or function from I^{q+1})

$$\left(f_{i_0,\dots,i_q}\right)_{i_0,\dots,i_q\in I} \qquad \text{with } f_{i_0,\dots,i_q} \in F\left(\bigcap_{j=0}^{q} U_{i_j}\right),$$

and $f_{i_0,\dots,i_q}$ is called its $(i_0,\dots,i_q)$-component.

We define a *coboundary operator* δ, carrying a q-cochain to a $(q+1)$-cochain.

Definition 7.10. For $q \in \mathbb{N}$, the **(q-th) coboundary operator** is the group homomorphism

$$\delta\colon C^q(U_i,F) \to C^{q+1}(U_i,F),$$

mapping a q-cochain $f \in C^q(U_i,F)$ to the $(q+1)$-cochain with $(i_0,\dots,i_q,i_{q+1})$-component

$$(\delta f)_{i_0,\dots,i_{q+1}} = \sum_{j=0}^{q+1}(-1)^j \, f_{i_0,\dots,\hat{i_j},\dots,i_{q+1}}\Big|_{U_{i_0}\cap\dots\cap U_{i_{q+1}}},$$

where the hat $\hat{\cdot}$ means that the respective index is omitted.

In particular, for $q=0$ the 0-th coboundary operator $\delta\colon C^0(U_i,F)\to C^1(U_i,F)$ maps $(f_i)_{i\in I}\in C^0(U_i,F)$ to the 1-chain with (i,j)-component $f_j|_{U_i\cap U_j} - f_i|_{U_i\cap U_j}$.
Similarly, the first coboundary operator $\delta\colon C^1(U_i,F) \to C^2(U_i,F)$ maps $(f_{i,j})_{i,j\in I} \in C^1(U_i,F)$ to the 2-chain with (i,j,k)-component $f_{j,k}|_{U_i\cap U_j\cap U_k} - f_{i,k}|_{U_i\cap U_j\cap U_k} + f_{i,j}|_{U_i\cap U_j\cap U_k}$.

A straightforward calculation establishes the following lemma.

Lemma 7.11. For any topological space X, open cover $(U_i)_{i\in I}$ of X and presheaf F on X, the abelian groups of q-chains with the coboundary operator δ form a *cochain complex*; that is, $\delta\circ\delta = 0$.

$$C^0(U_i,F) \xrightarrow{\ \delta\ } C^1(U_i,F) \xrightarrow{\ \delta\ } C^1(U_i,F) \xrightarrow{\ \delta\ } \dots$$

Definition 7.12. Let X be a topological space with an open cover $(U_i)_{i\in I}$ and F a presheaf on X. The **group of q-cocycles** is

$$Z^q(U_i,F) \coloneqq \ker\big(\delta\colon C^q(U_i,F)\to C^{q+1}(U_i,F)\big) \subset C^q(U_i,F).$$

The **group of q-coboundaries** is

$$B^q(U_i,F) \coloneqq \operatorname{im}\big(\delta\colon C^{q-1}(U_i,F)\to C^q(U_i,F)\big) \subset C^q(U_i,F).$$

By Lemma 7.11, we have $B^q(U_i,F)\subset Z^q(U_i,F)$, so we may define the **n-th (Čech) cohomology group (with coefficients in F relative to (U_i))** as the quotient group

$$H^q(U_i,F) \coloneqq Z^n(U_i,F)/B^n(U_i,F).$$

One may wonder whether the cohomology group depends on the chosen cover $(U_i)_{i\in I}$ of the topological space X. In general, this is indeed the case and in order to make it independent from the chosen cover, one can use a direct limit. However, we will content ourselves with the above definition, which then generally depends on the chosen cover $(U_i)_{i\in I}$ as indicated by the notation $H^q(U_i, F)$.

A 0-chain $(f_i)_{i\in I}$ satisfies $(f_i) \in Z^0(U_i, F)$ if and only if $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$. Since $B^0(U_i, F)$ is implicitly understood to be 0 (one can freely attach zero maps to the left of the cochain complex), $Z^0(U_i, F)$ is (isomorphic to) the zeroth cohomology group. If F is a sheaf, then we can glue the f_i to a global section $f \in F(X)$ with $f|_{U_i} = f_i$ for all $i \in I$. This construction yields a group isomorphism $Z^0(U_i, F) \cong F(X)$.
We summarize this result.

Proposition 7.13. Let X be a topological space with an open cover $(U_i)_{i\in I}$ and F a sheaf on X. Then $Z^0(U_i, F) \cong F(X)$, so in particular the zeroth cohomology group is independent of the chosen cover $(U_i)_{i\in I}$.

We are only interested in the first cohomology group $H^1(U_i, F)$.
Let $(f_{i,j})_{i,j\in I} \in C^1(U_i, F)$. Then $(f_{i,j}) \in Z^1(U_i, F)$ if and only if it satisfies the *cocycle condition*

$$f_{i,k} = f_{i,j} + f_{j,k} \qquad \text{on } U_i \cap U_j \cap U_k \quad \forall\ i, j, k \in I.$$

In particular, this implies that $f_{i,i} = 0$ and $f_{i,j} = -f_{j,i}$.
Similarly, $(f_{i,j}) \in B^1(U_i, F)$ if and only if there exists a 0-cochain $(g_i) \in C^0(U_i, F)$, such that $f_{i,j} = g_j|_{U_i\cap U_j} - g_i|_{U_i\cap U_j}$ for all $i, j \in I$.

Our next goal is to show that the first homology group with coefficients in the sheaf C^∞ of C^∞-functions is zero on any Riemann surface. Here being a C^∞-function is defined just like for ordinary smooth manifolds.
For this, we require the following lemma.

Lemma 7.14 (Existence of Partitions of Unity).
On every complex manifold X with an arbitrary open cover $(U_i)_{i\in I}$, there exists a *(smooth) partition of unity*; i.e. functions $(\psi_i)_{i\in I} \in C^\infty(X, \mathbb{R})$, such that for all $i \in I$, we have $\operatorname{im}(\psi_i) \in [0, 1]$, $\operatorname{supp}(\psi_i) \subset U_i$, $\sum_{j\in I} \psi_j = 1$ and so that every point $x \in X$ admits a neighborhood intersecting $\operatorname{supp}(\psi_i)$ for only finitely many $i \in I$.

Theorem 7.15. For a Riemann surface X, the first cohomology group $H^1(U_i, C^\infty)$ of the sheaf of smooth functions C^∞ is zero for any open cover $(U_i)_{i\in I}$.

Proof. Let $(f_{i,j})_{i,j\in I} \in Z^1(U_i, C^\infty)$ be a 1-cocycle. We have to construct $(g_i)_{i\in I} \in C^0(U_i, C^\infty)$ with $\delta(g_i) = (f_{i,j})$. By Lemma 7.14, there exists a partition of unity $(\psi_i)_{i\in I}$ on X. Then $\psi_j \cdot f_{i,j}$ can be interpreted as a smooth function on U_i (it is zero on $U_i \setminus U_j$, where $f_{i,j}$ is not defined). It follows that the function $g_i := \sum_{k\in I} \psi_k f_{k,i}$ is well-defined (plugging in any point yields a finite sum) and satisfies $g_i \in C^\infty(U_i, \mathbb{R})$. Thus the claim follows from the calculation (using the cocycle condition)

$$\begin{aligned} g_j|_{U_i\cap U_j} - g_i|_{U_i\cap U_j} &= \sum_{k\in I} \psi_k f_{k,j}|_{U_i\cap U_j} - \sum_{k\in I} \psi_k f_{k,i}|_{U_i\cap U_j} \\ &= \sum_{k\in I} \psi_k (f_{k,j} - f_{k,i})|_{U_i\cap U_j} = \sum_{k\in I} \psi_k f_{i,j} = f_{i,j}. \end{aligned}$$

□

To show that the first cohomology group with coefficients in the sheaf $\mathcal{O}$ of holomorphic functions is zero on $\mathbb{C}$, we need another lemma.

Lemma 7.16 (Dolbeault). For any $h \in C^\infty(\mathbb{C}, \mathbb{C})$, there exists $g \in C^\infty(\mathbb{C}, \mathbb{C})$, such that $h = \partial_{\bar{z}} g$.

Theorem 7.17. For the complex plane $\mathbb{C}$, the first cohomology group $H^1(U_i, \mathcal{O})$ of the sheaf of holomorphic functions $\mathcal{O}$ is zero for any open cover $(U_i)_{i \in I}$.

Proof. Let $(f_{i,j})_{i,j \in I} \in Z^1(U_i, \mathcal{O})$. Because $Z^1(U_i, \mathcal{O}) \subset Z^1(U_i, C^\infty)$, Theorem 7.15 yields the existence of a 0-cochain $(g_i)_{i \in I} \in C^0(U_i, C^\infty)$ satisfying $f_{i,j} = g_j|_{U_i \cap U_j} - g_i|_{U_i \cap U_j}$ for all $i, j \in I$. Because $f_{i,j} \in \mathcal{O}(U_i \cap U_j)$, we have $\partial_{\bar{z}} f_{i,j} = 0$ and thus $\partial_{\bar{z}} g_i = \partial_{\bar{z}} g_j$ on $U_i \cap U_j$. Therefore, there exists $h \in C^\infty(\mathbb{C}, \mathbb{C})$, such that $h|_{U_i} = \partial_{\bar{z}} g_i$ for all $i \in I$.
By Lemma 7.16, we can find $g \in C^\infty(\mathbb{C}, \mathbb{C})$, such that $h = \partial_{\bar{z}} g$. Then $f_i := g_i - g|_{U_i} \in C^\infty(U_i, \mathbb{C})$ satisfies $\partial_{\bar{z}} f_i = 0$, so it lies in $\mathcal{O}(U_i)$. Finally, the observation $f_j|_{U_i \cap U_j} - f_i|_{U_i \cap U_j} = f_{i,j}$ shows that $\delta((f_i)) = (f_{i,j})$. □

Finally, we turn to the sheaf of meromorphic functions $\mathcal{M}$.

Definition 7.18. Let X be a Riemann surface with open cover (U_i). A 0-cochain $(f_i)_{i \in I} \in C^0(U_i, \mathcal{M})$ is called a **Mittag-Leffler distribution**, if $\delta((f_i))$ is a 1-cochain in the sheaf of holomorphic functions $\mathcal{O}$; i.e. if $f_i - f_j \in \mathcal{O}(U_i \cap U_j)$ for all $i, j \in I$.
A **solution of a Mittag-Leffler distribution** $(f_i)_{i \in I}$ is a meromorphic function $f \in \mathcal{M}(X)$, such that $f|_{U_i} - f_i \in \mathcal{O}(U_i)$ for all $i \in I$.

By definition, $\delta((f_i)) = 0 \in H^1(U_i, \mathcal{M})$ for any 0-cochain $(f_i)_{i \in I} \in C^0(U_i, \mathcal{M})$. However, if $(f_i)_{i \in I}$ is a Mittag-Leffler distribution, then $\delta((f_i)) \in C^1(U_i, \mathcal{O})$ and we can ask whether it is zero in the corresponding cohomology group $H^1(U_i, \mathcal{O})$. This turns out to be equivalent to the Mittag-Leffler distribution $(f_i)_{i \in I}$ having a solution.

Theorem 7.19. Let X be a Riemann surface with open cover (U_i). A Mittag-Leffler distribution $(f_i)_{i \in I}$ has a solution if and only if $\delta((f_i)) = 0 \in H^1(U_i, \mathcal{O})$.

Proof. If $f \in \mathcal{M}(X)$ is a solution of $(f_i)_{i \in I}$ then $g_i := f_i - f|_{U_i} \in \mathcal{O}(U_i)$ satisfies $g_j - g_i = f_j - f_i$ on $U_i \cap U_j$, showing that $\delta((g_i)) = \delta((f_i)) \in C^1(U_i, \mathcal{O})$.
On the other hand, assume that $\delta((f_i)) = 0 \in H^1(U_i, \mathcal{O})$; that is, there exists a 0-cochain $(g_i)_{i \in I} \in C^0(U_i, \mathcal{O})$, such that $f_j - f_i = g_j - g_i$ on $U_i \cap U_j$ for all $i, j \in I$. In other words, we have $f_i - g_i = f_j - g_j$ on $U_i \cap U_j$ for all $i, j \in I$. Because $\mathcal{M}$ is a sheaf, these meromorphic functions glue to a global meromorphic function $f \in \mathcal{M}(X)$, such that $f|_{U_i} = f_i - g_i$ for all $i \in I$ and this f is a solution of the Mittag-Leffler distribution $(f_i)_{i \in I}$. □

This theorem together with Theorem 7.17 shows that every Mittag-Leffler distribution on $\mathbb{C}$ has a solution and one can show that the same holds true for the Riemann sphere $\hat{\mathbb{C}}$.

References

[Bor16] Folkmar Bornemann. *Funktionentheorie*. 2nd ed. Mathematik Kompakt. Birkhäuser Basel, 2016. ISBN: 978-3-0348-0973-3.

[FL94] Wolfgang Fischer and Ingo Lieb. *Funktionentheorie*. 7th ed. Wiesbaden: Vieweg + Teubner Verlag, 1994. X, 275. ISBN: 978-3-528-67247-8.

[For81] Otto Forster. *Lectures on Riemann Surfaces*. 1st ed. Vol. 81. Graduate Texts in Mathematics. New York, NY: Springer, 1981. VIII, 256. ISBN: 978-1-4612-5963-3.

[JS87] Gareth A. Jones and David Singerman. *Complex Functions: An Algebraic and Geometric Viewpoint*. 1st edition. Cambridge Cambridgeshire ; New York: Cambridge University Press, Mar. 27, 1987. 360 pp. ISBN: 978-0-521-31366-7.

[RS07] Reinhold Remmert and Georg Schumacher. *Funktionentheorie 2*. 3rd ed. Springer-Lehrbuch. Berlin, Heidelberg: Springer, 2007. ISBN: 978-3-540-40432-3.