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Change of Enrichment over monoidal 2-Categories

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Thesis for the attainment of the academic degree

Master of Science

at the TUM School of Computation, Information and Technology of the Technical University of Munich

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Submitted:
Munich, 16 August 2024

I hereby declare that this thesis is entirely the result of my own work except where otherwise indicated. I have only used the resources given in the list of references.

Munich, 16 August 2024 Marvin Jahn

Zusammenfassung

Das Wechseln der anreichernden monoidalen Kategorie spielt eine wichtige Rolle, um den Zusammenhang zwischen Kategorien, die über verschiedenen monoidalen Kategorien angereichert sind, zu verstehen. Diese Arbeit beschäftigt sich mit diesem Wechsel für Kategorien, welche über monoidalen (strikten) 2-Kategorien angereichert sind. Wir zeigen, dass die entsprechende Konstruktion wohldefiniert ist. Außerdem beweisen wir, dass dies einen strikten 2-Funktor von der Kategorie der (kleinen) monoidalen 2-Kategorien zur Kategorie der (kleinen) Kategorien darstellt. Danach betrachten wir dargestellte 2-Funktoren $\mathcal{V}(C,-)\colon \mathcal{V} \to \mathcal{C}at$ und zeigen, dass die Struktur, welche den Wechsel der anreichernden monoidalen Kategorie entlang dieses Funktors erlaubt, genau der eines Komonoids auf dem darstellenden Objekt C entspricht. Als Korollar sehen wir, dass eine kanonische natürliche Transformation $(-)_0 \Rightarrow \mathrm{ChEn}(-,\mathcal{F})$ von dem zugrunde liegenden Bikategorienfunktor $(-)_0$ zu dem “Wechsel der Anreicherung” Funktor existiert. Schließlich beschreiben wir, wie eine erweiterte Version der Bikategorie der Spanne aus derselbigen formal konstruiert werden kann, indem man in geeigneter Art und Weise die anreichernde monoidale Kategorie wechselt.

Abstract

Change of enrichment plays an important role in understanding the relationship between categories enriched over distinct monoidal categories. In this thesis, we consider change of enrichment for categories enriched over monoidal (strict) 2-categories and show that the corresponding construction is well-defined. Furthermore, we prove that it yields a strict 2-functor from the category of (small) monoidal 2-categories to the category of (small) categories. We then consider represented 2-functors $\mathcal{V}(C,-)\colon \mathcal{V} \to \mathcal{C}at$ and demonstrate that the structure needed to change enrichment along these functors is equivalent to the structure of a comonoid on the representing object C. As a corollary, we see that there is a canonical natural transformation $(-)_0 \Rightarrow \mathrm{ChEn}(-,\mathcal{F})$ from the underlying bicategory functor $(-)_0$ to the functor that changes the enrichment along a fixed functor $\mathcal{F}$. Finally, we describe how an extended version of the bicategory of spans can be formally constructed via change of enrichment.

Contents

1 Introduction

It is a common phenomenon that categories arising in a mathematical theory are not just conventional categories, but admit additional structure on their "hom-sets". A well-known example is the $(\mathcal{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$-enriched category $\mathcal{Mod}_R$ of modules over a fixed ring R, where $(\mathcal{Ab}, \otimes_{\mathbb{Z}}, \mathbb{Z})$ is the monoidal category of abelian groups equipped with the tensor product $\otimes_{\mathbb{Z}}$ as its monoidal product. The fact that any closed symmetric monoidal category (in particular any cartesian closed one) is enriched over itself yields many more examples, like the monoidal category of vector spaces $(\mathcal{Vect}_{\mathbb{K}}, \otimes_{\mathbb{K}}, \mathbb{K})$ over a field $\mathbb{K}$, the cartesian monoidal category of (small) categories $(\mathcal{Cat}, \times, \{*\})$ or a convenient subcategory (see [Ste67]) of the category of topological spaces $\mathcal{Top}$.

Originally conceived independently by Bénabou and Maranda [Bén65; Mar65], the language of enriched category theory captures these and many more examples. However, in many situations, it is necessary or convenient to change the monoidal category over which one considers enrichments in. This *change of enrichment* along a lax monoidal functor $(\mathcal{V}, \otimes, \mathbb{1}) \to (\mathcal{W}, \otimes, \mathbb{1})$ was first considered in [EK66]. Because such a monoidal category is also called *base of enrichment*, it is common to refer to this procedure as *change of base*, but in order to avoid ambiguity, we will exclusively refer to it as *change of enrichment*.

A trivial example of this is that a category enriched over any monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$ should have an underlying ordinary category (i.e. a $(\mathcal{Set}, \times, \{*\})$-enriched category). This idea is made precise by changing the enrichment from $\mathcal{V}$ to $\mathcal{Set}$ (see Example 2.1.2). A more sophisticated instance emerges in the theory of $(\infty,1)$-categories, where both topologically and simplicially enriched categories are considered as models for higher categories. In this case, change of enrichment provides a way to translate between these two points of view (see Example 2.1.5).

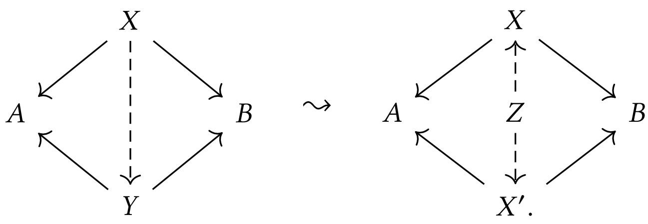
In this thesis, we study change of enrichment for categories enriched over monoidal (strict) 2-categories. Our main source of motivation is the case of the cartesian monoidal 2-category $(\mathcal{Cat}, \times, \{*\})$. Because a category enriched over the cartesian monoidal 2-category $(\mathcal{Cat}, \times, \{*\})$ is a bicategory, such a theory in particular yields a way to systematically obtain bicategories from other bicategories by "replacing their hom-categories" (see Example 2.4.2). Since we can view any monoidal category as a monoidal 2-category with only identity 2-cells, enrichment over monoidal 2-categories encompasses enrichment over ordinary categories as a special case.

Chapter 2 starts by recalling change of enrichment for conventional categories and collecting some of the basic concepts of bicategory theory that we will require. We then proceed to define monoidal 2-categories and the functors between monoidal 2-categories along which change of enrichment can be performed. Since these were the lax monoidal functors in the case of enrichment over ordinary categories, it is not too surprising that the corresponding notion for monoidal 2-categories is that of a lax monoidal functor of 2-categories (which we call, more suggestively, *changer*). Having established the required terminology, we perform our main construction, namely Construction 2.4.1. We finish the chapter by describing (see Theorem 2.6.2) change of enrichment over monoidal 2-categories as the action on hom-category of a strict 2-functor

$$\mathrm{En}\colon \mathcal{2Cat}_{\otimes}^{lax} \to \mathcal{Cat},\ \mathcal{V} \mapsto \mathcal{V}\text{-}\mathcal{Cat}.$$

In Chapter 3, we highlight a particular class of changers, namely the changers $\mathcal{V}(C, -)\colon \mathcal{V} \to \mathcal{Cat}$ represented by a fixed object $C \in \mathcal{V}$, where $(\mathcal{V}, \otimes, \mathbb{1})$ is a monoidal (strict) 2-category. Establishing a correspondence between represented changers and comonoids (see Theorem 3.1.3) in particular yields some interesting examples of change of enrichment over monoidal 2-categories. For instance, the aforementioned underlying category construction extends to an underlying bicategory functor when enriching over monoidal 2-categories.

In the final Chapter 4, we apply the developed theory to the bicategory of spans, producing an extended bicategory of spans, essentially by formally replacing the 2-cells:



Such iterated spans are of interest in the context of extended topological field theories (see [Hau18, Sec 1.2]) and our application can be seen as the "bicategorical version" of this idea. After defining the category and bicategory of spans, we formally derive the extended bicategory of spans (see Definition 4.3.7) from the ordinary one via change of enrichment along the *spanification functor* from Definition 4.3.3. We finish by demonstrating how some of the properties of the bicategory of spans directly translate to the extended bicategory due to the functoriality of change of enrichment. Understanding how this construction can be perceived as a special case of change of enrichment served as one of the author's primary motivations for developing the theory presented in this text.

2 Change of Enrichment for Categories enriched over monoidal 2-Categories

We start by describing change of enrichment over ordinary monoidal categories, which will serve as inspiration for our generalization to enrichment over monoidal 2-categories. After establishing the main construction (Construction 2.4.1), we use the remaining part of the chapter to prove the functoriality of change of enrichment, culminating in Theorem 2.6.2.

2.1 Change of Enrichment for ordinary enriched Categories

It is a basic observation (first studied in [EK66, Prop 6.3] and reviewed e.g. in [Rie14, Lem 3.4.3]) that a lax monoidal functor $F\colon \mathcal{V} \to \mathcal{W}$ between two monoidal categories $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$ induces a canonical way to transform a $\mathcal{V}$-enriched category into a $\mathcal{W}$-enriched one. This process is known as *change of enrichment* (or *change of base*).

Construction 2.1.1 (Change of Enrichment). Let $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$ be two monoidal categories and $F\colon \mathcal{V} \to \mathcal{W}$ be a lax monoidal functor between them. Then a category $C_{\mathcal{V}}$ enriched over $\mathcal{V}$ induces a category $C_{\mathcal{W}}$ enriched over $\mathcal{W}$ as follows:

1. The objects are those of $C_{\mathcal{V}}$ and the hom-objects are induced by F: $C_{\mathcal{W}}(X, Y) \coloneqq F(C_{\mathcal{V}}(X, Y))$.
2. The unit is given by the composition

$$\mathbb{1} \longrightarrow F(\mathbb{1}) \xrightarrow{F(\mathrm{id}_X)} F(C_{\mathcal{V}}(X, X)).$$

3. The composition morphism in $C_{\mathcal{W}}$ is defined to be

$$F(C_{\mathcal{V}}(Y, Z)) \otimes F(C_{\mathcal{V}}(X, Y)) \longrightarrow F(C_{\mathcal{V}}(Y, Z) \otimes C_{\mathcal{V}}(X, Y)) \xrightarrow{F(\circ)} F(C_{\mathcal{V}}(X, Z)).$$

The construction, while quite straightforward, is surprisingly useful.

Example 2.1.2. For instance, changing the enrichment allows us to extract the *underlying category* of a category enriched over a monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$ using the functor $F = \mathrm{Hom}(\mathbb{1}, -)\colon \mathcal{V} \to \mathcal{S}et$ represented by the unit object $\mathbb{1}$. See Definition 3.2.2 for an extension of this idea.

An important property is that change of enrichment can be viewed as a 2-functor, which is shown in [Cru08, Thm 4.3.2]. Before stating the theorem, we introduce the necessary categories.

Definition 2.1.3. We define the following categories.

1. $\mathcal{C}at$ is the 2-category consisting of the (small) categories, functors and natural transformations.
2. $2\mathcal{C}at$ denotes the 2-category of (small) 2-categories, 2-functors and 2-natural transformations.
3. $\mathcal{C}at_{\otimes}^{lax}$ is the 2-category of (small) monoidal categories, lax monoidal functors and monoidal natural transformations.
4. For a monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$, the 2-category $\mathcal{V}\text{-}\mathcal{C}at$ consists of the (small) $\mathcal{V}$-enriched categories, the $\mathcal{V}$-enriched functors and the $\mathcal{V}$-enriched natural transformations.

Theorem 2.1.4. Change of enrichment extends to a 2-functor

$$\mathcal{Cat}_{\otimes}^{lax} \to 2\mathcal{Cat},\ \mathcal{V} \mapsto \mathcal{V}\text{-}\mathcal{Cat}.$$

Our version of this theorem for categories enriched over monoidal 2-categories is Theorem 2.6.2. An immediate consequence of the previous theorem is that adjunctions are preserved, giving rise to multiple applications.

Example 2.1.5.

1. Let $\mathcal{Top}$ denote the category of compactly generated Hausdorff spaces and $s\mathcal{Set}$ the category of simplicial sets. The adjunction

$$s\mathcal{Set} \underset{\text{Sing}}{\overset{|\cdot|}{\rightleftarrows}} \mathcal{Top} \quad (\bot)$$

   between geometric realization $|\cdot|\colon s\mathcal{Set} \to \mathcal{Top}$ and its right adjoint, the total singular complex functor Sing: $\mathcal{Top} \to s\mathcal{Set}$, is of fundamental importance in the theory of $(\infty,1)$-categories. Because both functors preserve finite products and are thus strong monoidal functors (in particular lax monoidal), this induces by functoriality of change of enrichment an adjunction between the category of simplicially enriched categories and the topologically enriched ones[1]. This adjunction characterizes the relationship between the theory of simplicially enriched categories and that of topologically enriched categories, both of which have been considered as models for higher category theory [Lur09, Rem 1.1.4.3].

2. The nerve functor $N\colon \mathcal{Cat} \hookrightarrow s\mathcal{Set}$ admits a left adjoint, which is the homotopy category functor $h\colon s\mathcal{Set} \to \mathcal{Cat}$, obtained via left Kan extension of $\Delta \to \mathcal{Cat}$ along the Yoneda embedding $\Delta \hookrightarrow s\mathcal{Set}$

$$s\mathcal{Set} \underset{N}{\overset{h}{\rightleftarrows}} \mathcal{Cat}. \quad (\bot)$$

   As both of these functors preserve finite products, we obtain an adjunction between categories enriched over simplicial sets and categories enriched over $\mathcal{Cat}$ (i.e. strict 2-categories) [RV22, Exa A.7.10].

3. The localization functor $s\mathcal{Set} \to \mathrm{Ho}(s\mathcal{Set})$ into the homotopy category of spaces $\mathrm{Ho}(s\mathcal{Set})$ can be shown to be lax monoidal (see [Rie14, Cha 10]), so any simplicially enriched category induces a category enriched over $\mathrm{Ho}(s\mathcal{Set})$. This observation can be used to define DK-equivalences as those simplicial functors whose induced $\mathrm{Ho}(s\mathcal{Set})$-enriched functor is an equivalence of $\mathrm{Ho}(s\mathcal{Set})$-enriched categories.

2.2 Bicategorical Preliminaries

As a first step towards categories enriched over monoidal 2-categories, we recall the notion of a bicategory. This concept, originally introduced by Bénabou in 1967 in [Bén+67], was the first precise notion of a weak higher category; that is, a category-like structure with morphisms between morphisms for which relations like associativity are only required up to coherent isomorphism.

Definition 2.2.1. A **bicategory** C consists of the following data:

1. A collection $\mathrm{Ob}(C)$ whose elements are called *objects* or 0-*cells*.

[1] It is common to speak of topologically enriched categories even if one only enriches over a convenient subcategory.

2. For each pair $X, Y \in \mathrm{Ob}(C)$, a category $C(X, Y)$, called *hom-category*. Its objects are called 1-*cells* and the morphisms are called 2-*cells*. The composition of morphisms in this category is called *vertical composition*.

3. For each $X \in \mathrm{Ob}(C)$, a special 1-cell $\mathrm{id}_X \in C(X, X)$, called the *identity* 1-*cell* of X.

4. For each triple $X, Y, Z \in \mathrm{Ob}(C)$, a functor

$$\bullet\colon C(Y, Z) \times C(X, Y) \to C(X, Z),$$

which encodes the *horizontal composition* of morphisms. The action of this functor on morphisms (2-cells) is usually denoted by $*$.

5. For every $X, Y \in \mathrm{Ob}(C)$, two natural isomorphisms

$$\rho\colon\ - \bullet \mathrm{id}_X \overset{\cong}{\Longrightarrow} \mathrm{id}_{C(X,Y)}, \quad \lambda\colon\ \mathrm{id}_Y \bullet - \overset{\cong}{\Longrightarrow} \mathrm{id}_{C(X,Y)}.$$

λ is called *left unitor* and ρ *right unitor*.

6. For all $W, X, Y, Z \in \mathrm{Ob}(C)$, a natural isomorphism α (called *associator*):

$$\begin{array}{ccccc}
(C(Y,Z) \times C(X,Y)) \times C(W,X) & \xrightarrow{\bullet \times \mathrm{id}_{C(W,X)}} & C(X,Z) \times C(W,X) & \xrightarrow{\bullet} & \\
\downarrow{\cong} & \Downarrow{\alpha} & & & C(W,Z) \\
C(Y,Z) \times (C(X,Y) \times C(W,X)) & \xrightarrow{\mathrm{id}_{C(Y,Z)} \times \bullet} & C(Y,Z) \times C(W,Y). & \xrightarrow{\bullet} &
\end{array}$$

This data must make the following diagrams commute for all 1-cells f, g, h, k:

1. *Pentagon identity*:

$$\begin{array}{ccccc}
 & & (k \bullet h) \bullet (g \bullet f) & & \\
 & \nearrow{\alpha} & & \searrow{\alpha} & \\
((k \bullet h) \bullet g) \bullet f & & & & k \bullet (h \bullet (g \bullet f)) \\
\downarrow{\alpha * \mathrm{id}_f} & & & & \uparrow{\mathrm{id}_k * \alpha} \\
(k \bullet (h \bullet g)) \bullet f & & \xrightarrow{\alpha} & & k \bullet ((h \bullet g) \bullet f);
\end{array}$$

2. *Triangle identity*:

$$\begin{array}{ccc}
(g \bullet \mathrm{id}) \bullet f & \xrightarrow{\alpha} & g \bullet (\mathrm{id} \bullet f) \\
& {\scriptstyle \rho * \mathrm{id}_f} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_g * \lambda} & \\
& g \bullet f. &
\end{array}$$

A bicategory for which λ, ρ and α are identities is called a **(strict) 2-category**.[2]

Example 2.2.2.

1. A 2-category can be equivalently described as a $\mathcal{C}at$-enriched category, where $\mathcal{C}at$ denotes the category of (small) categories. The archetypal example of a 2-category is $\mathcal{C}at$ itself. Its objects are the (small) categories, its 1-cells are functors and its 2-cells are natural transformations.

[2] For us, a 2-category is strict by default and we call the weak version a *bicategory*. Note that parts of the literature refer to *bicategories* as 2-*categories* and to 2-*categories* as *strict* 2-*categories*.

2. A bicategory C with only one object $*$ is the same as a monoidal category. Here the category underlying the monoidal category is defined to be $C(*,*)$ and the monoidal product is given by $\bullet$. This process of viewing a monoidal category as a bicategory with one object is called *delooping*.

3. Any ordinary category can be viewed as a 2-category whose only 2-cells are identities.

4. A more sophisticated example is given by the bicategory of spans, which we formally construct in Definition 4.2.6.

The following bracket notation for functors induced by the universal property of the product in $\mathcal{C}at$ will prove useful.

Notation 2.2.3. For $i \in I$, let $L_i \colon \mathcal{D} \to \mathcal{E}_i$ be a functor. We write

$$[L_i]_{i\in I} \colon \mathcal{D} \to \prod_{i\in I} \mathcal{E}_i$$

for the unique functor satisfying $\pi_j \circ [L_i]_{i\in I} = L_j$ for all $j \in I$. In other words, this is just the functor induced by the L_i from the universal property of the product in $\mathcal{C}at$.
Furthermore, natural transformations

$$\mathcal{D} \underset{L_i}{\overset{K_i}{\rightrightarrows}} \mathcal{E}_i \;\; (\Downarrow \alpha_i) \quad \rightsquigarrow \quad \mathcal{D} \underset{[L_i]}{\overset{[K_i]}{\rightrightarrows}} \textstyle\prod_{i\in I} \mathcal{E}_i \;\; (\Downarrow [\alpha_i])$$

as on the left for every $i \in I$, induce a natural transformation as on the right, characterized by the property that $\pi_j \circ [\alpha_i]_{i\in I} = \alpha_j$ for all $j \in I$. For a fixed index set I and categories $\mathcal{D}$, $\{\mathcal{E}_i\}_{i\in I}$, this constitutes a functor

$$[-]_{i\in I} \colon \prod_{i\in I} [\mathcal{D}, \mathcal{E}_i] \to \left[\mathcal{D}, \prod_{i\in I} \mathcal{E}_i\right].$$

We also introduce a particular notation for constant functors.

Notation 2.2.4. For two categories $\mathcal{B}$ and C and an object $X \in C$, we write $\mathrm{const}(X) \colon \mathcal{B} \to C$ for the constant functor picking the object $X \in C$ and the identity id_X. Moreover, for a morphism $f \colon X \to Y$, we denote the constant natural transformation by $\mathrm{const}(f) \colon \mathrm{const}(X) \Rightarrow \mathrm{const}(Y)$.

An important observation is that the pentagon and triangle identity of a bicategory can be stated purely in terms of functors and natural transformations. This helps uncover the fact that a bicategory is just a category suitably enriched over the monoidal 2-category $(\mathcal{C}at, \times, \{*\})$, as we realize in Example 2.3.5.

Remark 2.2.5. The pentagon and triangle identity for a bicategory are equivalent to the commutativity of the following diagrams:

1. *Pentagon identity*:

$$\begin{array}{ccccc}
 & & (- \bullet -) \bullet (- \bullet -) & & \\
 & \overset{\alpha\circ(\bullet\times\mathrm{id}\times\mathrm{id})}{\nearrow} & & \overset{\alpha\circ(\mathrm{id}\times\mathrm{id}\times\bullet)}{\searrow} & \\
((- \bullet -) \bullet -) \bullet - & & & & - \bullet (- \bullet (- \bullet -)) \\
\Downarrow {\scriptstyle \bullet\circ(\alpha\times\mathrm{id}_{\mathrm{id}})} & & & & \Uparrow {\scriptstyle \bullet\circ(\mathrm{id}_{\mathrm{id}}\times\alpha)} \\
(- \bullet (- \bullet -)) \bullet - & & \xRightarrow{\alpha\circ(\mathrm{id}\times\bullet\times\mathrm{id})} & & - \bullet ((- \bullet -) \bullet -).
\end{array}$$

This is a diagram in the category of functors $C(Y,Z) \times C(X,Y) \times C(W,X) \times C(V,W) \to C(V,Z)$ for objects $V, W, X, Y, Z \in \mathrm{Ob}(C)$.

2. *Triangle identity*:

$$\begin{array}{ccc} (- \bullet \mathrm{id}) \bullet - & \xRightarrow{\alpha \circ (\mathrm{id} \times [\mathrm{const}(\mathrm{id}), \mathrm{id}])} & - \bullet (\mathrm{id} \bullet -) \\ & {\scriptstyle \bullet \circ (\rho \times \mathrm{id}_{\mathrm{id}})} \searrow \quad \swarrow {\scriptstyle \bullet \circ (\mathrm{id}_{\mathrm{id}} \times \lambda)} & \\ & \bullet. & \end{array}$$

This is a diagram in the category of functors $C(Y,Z) \times C(X,Y) \to C(X,Z)$ for objects $X, Y, Z \in \mathrm{Ob}(C)$. We have used Notation 2.2.3 and Notation 2.2.4 and denoted the identity natural transformation of the identity functor by $\mathrm{id}_{\mathrm{id}}$. ○

There exist multiple appropriate notions of morphisms between bicategories.

Definition 2.2.6. A **lax functor** $F : C \to \mathcal{D}$ between two bicategories C and $\mathcal{D}$ is the following data:

1. A function $F : \mathrm{Ob}(C) \to \mathrm{Ob}(\mathcal{D})$ defining the action on objects.
2. For each pair of objects $X, Y \in \mathrm{Ob}(C)$, a functor
$$F = F(X,Y) : C(X,Y) \to \mathcal{D}(F(X), F(Y)).$$
3. For every object $X \in \mathrm{Ob}(C)$, a 2-cell $F_{\mathrm{id}} : \mathrm{id}_{F(X)} \Rightarrow F(\mathrm{id}_X)$, called *unity constraint.*
4. For every triple of objects $X, Y, Z \in \mathrm{Ob}(C)$, a natural transformation
$$\begin{array}{ccc} C(Y,Z) \times C(X,Y) & \xrightarrow{F \times F} & \mathcal{D}(F(Y),F(Z)) \times \mathcal{D}(F(X),F(Y)) \\ \downarrow \bullet & \overset{F_\bullet}{\Longleftarrow} & \bullet \downarrow \\ C(X,Z) & \xrightarrow{F} & \mathcal{D}(F(X),F(Z)) \end{array}$$
called *functoriality constraint.*

It must make the following three diagrams commute (for 1-cells $f \in C(W,X)$, $g \in C(X,Y)$, $h \in C(Y,Z)$):

1. *Associativity*:
$$\begin{array}{ccc} (F(h) \bullet F(g)) \bullet F(f) & \xrightarrow{\cong} & F(h) \bullet (F(g) \bullet F(f)) \\ {\scriptstyle F_\bullet * \mathrm{id}_{F(f)}} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{F(h)} * F_\bullet} \\ F(h \bullet g) \bullet F(f) & & F(h) \bullet F(g \bullet f) \\ {\scriptstyle F_\bullet} \downarrow & & \downarrow {\scriptstyle F_\bullet} \\ F((h \bullet g) \bullet f) & \xrightarrow{F(\cong)} & F(h \bullet (g \bullet f)). \end{array}$$
2. *Unity*:
$$\begin{array}{ccc} \mathrm{id}_{F(X)} \bullet F(f) & \xrightarrow{\lambda} & F(f) \\ {\scriptstyle F_{\mathrm{id}} * \mathrm{id}_{F(f)}} \downarrow & & \uparrow {\scriptstyle F(\lambda)} \\ F(\mathrm{id}_X) \bullet F(f) & \xrightarrow[F_\bullet]{} & F(\mathrm{id}_X \bullet f) \end{array} \qquad \begin{array}{ccc} F(f) \bullet \mathrm{id}_{F(W)} & \xrightarrow{\rho} & F(f) \\ {\scriptstyle \mathrm{id}_{F(f)} * F_{\mathrm{id}}} \downarrow & & \uparrow {\scriptstyle F(\rho)} \\ F(f) \bullet F(\mathrm{id}_W) & \xrightarrow[F_\bullet]{} & F(f \bullet \mathrm{id}_W). \end{array}$$

Furthermore, if the unity and functoriality constraint are isomorphisms, then F is called a **functor (between bicategories)** or a **pseudofunctor**. If they are even the identity, then F is called a **strict functor**. In situations where confusion with the 1-dimensional setting may arise, we also say 2-**functor** instead of *functor.*

Just like the pentagon and triangle identity in the definition of a bicategory can be stated purely in terms of functors and natural transformations (see Remark 2.2.5), the same is true for the associativity and unity diagrams for functors between bicategories. This point of view allows generalizing functors between bicategories to functors between categories enriched over monoidal 2-categories (see Example 2.5.2).

Remark 2.2.7. The associativity and unity diagrams can equivalently be described as diagrams of functors and natural transformations:

1. *Associativity*:

$$\begin{array}{ccc}
(F(-) \bullet F(-)) \bullet F(-) & \xRightarrow{\alpha\circ(F\times F\times F)} & F(-) \bullet (F(-) \bullet F(-)) \\
{\scriptstyle \bullet\circ(F_\bullet\times \mathrm{id}_F)}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \bullet\circ(\mathrm{id}_F\times F_\bullet)} \\
F(- \bullet -) \bullet F(-) & & F(-) \bullet F(- \bullet -) \\
{\scriptstyle F_\bullet\circ(\bullet\times \mathrm{id})}\Big\Downarrow & & \Big\Downarrow{\scriptstyle F_\bullet\circ(\mathrm{id}\times\bullet)} \\
F((- \bullet -) \bullet -) & \xRightarrow{F\circ\alpha} & F(- \bullet (- \bullet -)).
\end{array}$$

This constitutes a diagram in the category of functors $\mathcal{C}(Y,Z)\times\mathcal{C}(X,Y)\times\mathcal{C}(W,X) \to \mathcal{D}(F(W),F(Z))$ for objects $W,X,Y,Z \in \mathrm{Ob}(\mathcal{C})$.

2. *Unity*:

$$\begin{array}{ccc}
\mathrm{id} \bullet F(-) & \xRightarrow{\lambda\circ F} & F \\
\Big\Downarrow{\scriptstyle \bullet\circ[\mathrm{const}(F_{\mathrm{id}}),\mathrm{id}_F]} & & \Big\Uparrow{\scriptstyle F\circ\lambda} \\
F(\mathrm{id}) \bullet F(-) & \xRightarrow[F_\bullet\circ[\mathrm{const}(\mathrm{id}),\mathrm{id}]]{} & F(\mathrm{id} \bullet -)
\end{array}
\qquad
\begin{array}{ccc}
F(-) \bullet \mathrm{id} & \xRightarrow{\rho\circ F} & F \\
\Big\Downarrow{\scriptstyle \bullet\circ[\mathrm{id}_F,\mathrm{const}(F_{\mathrm{id}})]} & & \Big\Uparrow{\scriptstyle F\circ\rho} \\
F(-) \bullet F(\mathrm{id}) & \xRightarrow[F_\bullet\circ[\mathrm{id},\mathrm{const}(\mathrm{id})]]{} & F(- \bullet \mathrm{id}).
\end{array}$$

Note that both of these diagrams live in the category of functors $\mathcal{C}(X,Y) \to \mathcal{D}(F(X),F(Y))$ for objects $X,Y \in \mathrm{Ob}(\mathcal{C})$ and we e.g. have

$$\mathrm{id} \bullet F(-) = \bullet \circ [\mathrm{const}(\mathrm{id}), F].$$ ○

With the lax functors as morphisms, the (small) bicategories form a category $\mathcal{Bicat}$ [Joh+21, Thm 4.1.30]. This also follows from the more general setting for categories and functors enriched over a monoidal 2-category (Lemma 2.5.3).

2.3 Monoidal 2-Categories and Changers

We introduce the concept of a monoidal 2-category.

Definition 2.3.1. A **monoidal 2-category** $(\mathcal{V}, \otimes, \mathbb{1})$ is a (strict) 2-category $\mathcal{V}$ together with

1. a (strict) 2-functor $\otimes\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ (called **monoidal product**);
2. an object $\mathbb{1} \in \mathrm{Ob}(\mathcal{V})$ (called **unit object**);
3. a 2-natural isomorphism $\alpha\colon (- \otimes -) \otimes - \Rightarrow - \otimes (- \otimes -)$ (called **associator**)
4. 2-natural isomorphisms $\lambda\colon \mathbb{1} \otimes - \Rightarrow \mathrm{id}$, $\rho\colon - \otimes \mathbb{1} \Rightarrow \mathrm{id}$ (called **left** and **right unitor**, respectively),

such that the underlying data (i.e. forgetting the 2-cells) $(\mathcal{V}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ forms a monoidal category.
A monoidal 2-category $(\mathcal{V}, \otimes, \mathbb{1})$ is called **cartesian**, if its monoidal product $\otimes$ is given by its 2-product ($\mathcal{Cat}$-enriched product) $\times\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V}$.
A **monoidal 2-subcategory** $(\mathcal{B}, \otimes|_{\mathcal{B}\times\mathcal{B}}, \mathbb{1}) \subset (\mathcal{V}, \otimes, \mathbb{1})$ consists of a 2-subcategory $\mathcal{B} \subset \mathcal{V}$, such that $\otimes$ restricts to a 2-functor $\otimes|_{\mathcal{B}\times\mathcal{B}} : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and such that $\mathbb{1} \in \mathcal{B}$.

Since a (strict) 2-category is the same as a $\mathcal{Cat}$-enriched category, we may alternatively regard a monoidal 2-category as a monoidal category which is $\mathcal{Cat}$-enriched in a suitable sense.

Remark 2.3.2. The definition of monoidal 2-categories is rather concise, since we are dealing with (strict) 2-categories instead of bicategories. The corresponding concept for bicategories is significantly more complicated due to the many coherence relations imposed, details can be found in [Sta16, Def 4.4] or for the case of semistrict monoidal 2-categories in [Cra98, Def 2.1]. A simpler notion of cartesian bicategories was introduced in [CW87, Def 1.2]. The more general idea of monoidal (∞,n)-categories has been gaining significant attention due to its appearance in the cobordism hypothesis [Lur08, Thm 1.4.9]. ○

Example 2.3.3.

1. The most important example of a (cartesian) monoidal 2-category is the category of small categories $(\mathcal{Cat}, \times, \{*\})$.
2. Any monoidal category can be viewed as a monoidal 2-category by considering it as a 2-category with no non-identity cells.

Inspired by the intuition that an ordinary bicategory should be a category enriched over the monoidal 2-category $(\mathcal{Cat}, \times, \{*\})$, we define the notion of categories enriched over a given monoidal 2-category. The definition essentially consists of combining the concept of an ordinary enriched category and a bicategory. A more general notion of a (bi)category enriched in a monoidal bicategory has been suggested in [GS16, 3.1].

Definition 2.3.4. Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal 2-category. A **$\mathcal{V}$-enriched category** C consists of the following data:

1. A collection $\mathrm{Ob}(C)$ whose elements are called *objects* or 0-*cells*.
2. For each pair $X, Y \in \mathrm{Ob}(C)$, an object $C(X,Y) \in \mathcal{V}$.
3. For every $X \in \mathrm{Ob}(C)$, a special 1-cell $\mathrm{id}_X^C \colon \mathbb{1} \to C(X,X)$ in $\mathcal{V}$, which is called the *identity* 1-*cell* of X.
4. For each triple $X, Y, Z \in \mathrm{Ob}(C)$, a 1-cell
$$\bullet\colon C(Y,Z) \otimes C(X,Y) \to C(X,Z)$$
in $\mathcal{V}$, called *horizontal composition.*
5. For every $X, Y \in \mathrm{Ob}(C)$, two invertible 2-cells
$$\rho\colon \Big(C(X,Y) \cong C(X,Y) \otimes \mathbb{1} \xrightarrow{\mathrm{id}_{C(X,Y)} \otimes \mathrm{id}_X^C} C(X,Y) \otimes C(X,X) \xrightarrow{\ \bullet\ } C(X,Y) \Big) \overset{\cong}{\Longrightarrow} \mathrm{id}_{C(X,Y)},$$
$$\lambda\colon \Big(C(X,Y) \cong \mathbb{1} \otimes C(X,Y) \xrightarrow{\mathrm{id}_Y^C \otimes \mathrm{id}_{C(X,Y)}} C(Y,Y) \otimes C(X,Y) \xrightarrow{\ \bullet\ } C(X,Y) \Big) \overset{\cong}{\Longrightarrow} \mathrm{id}_{C(X,Y)}.$$
λ is called *left unitor* and ρ *right unitor.*
6. For all $W, X, Y, Z \in \mathrm{Ob}(C)$, an invertible 2-cell α (called *associator*):
$$\begin{array}{ccccc}
(C(Y,Z) \otimes C(X,Y)) \otimes C(W,X) & \xrightarrow{\bullet \otimes \mathrm{id}_{C(W,X)}} & C(X,Z) \otimes C(W,X) & & \\
 & & & \searrow^{\bullet} & \\
\Big\downarrow \cong & \Big\Downarrow \alpha & & & C(W,Z) \\
 & & & \nearrow_{\bullet} & \\
C(Y,Z) \otimes (C(X,Y) \otimes C(W,X)) & \xrightarrow{\mathrm{id}_{C(Y,Z)} \otimes \bullet} & C(Y,Z) \otimes C(W,Y). & &
\end{array}$$

The following two diagrams must commute:

1. *Pentagon identity*:

$$\begin{array}{ccccc}
 & & \bullet \circ (\bullet \otimes \bullet) & & \\
 & \overset{\alpha\circ(\bullet\otimes\mathrm{id}\otimes\mathrm{id})}{\nearrow} & & \overset{\alpha\circ(\mathrm{id}\otimes\mathrm{id}\otimes\bullet)}{\searrow} & \\
\bullet \circ (\bullet \otimes \mathrm{id}) \circ (\bullet \otimes \mathrm{id} \otimes \mathrm{id}) & & & & \bullet \circ (\mathrm{id} \otimes \bullet) \circ (\mathrm{id} \otimes \mathrm{id} \otimes \bullet) \\
\Downarrow {\scriptstyle \bullet\circ(\alpha\otimes\mathrm{id}_{\mathrm{id}})} & & & & \Uparrow {\scriptstyle \bullet\circ(\mathrm{id}_{\mathrm{id}}\otimes\alpha)} \\
\bullet \circ (\bullet \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \bullet \otimes \mathrm{id}) & & \xRightarrow{\alpha\circ(\mathrm{id}\otimes\bullet\otimes\mathrm{id})} & & \bullet \circ (\mathrm{id} \otimes \bullet) \circ (\mathrm{id} \otimes \bullet \otimes \mathrm{id}).
\end{array}$$

   Note that $\mathrm{id}_{\mathrm{id}}$ refers to the identity 2-cell of an identity 1-cell in $\mathcal{V}$.

2. *Triangle identity*:

$$\begin{array}{ccc}
\bullet \circ (\bullet \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id}^C \otimes \mathrm{id}) & \xRightarrow{\alpha\circ(\mathrm{id}\otimes\mathrm{id}^C\otimes\mathrm{id})} & \bullet \circ (\mathrm{id} \otimes \bullet) \circ (\mathrm{id} \otimes \mathrm{id}^C \otimes \mathrm{id}) \\
 \underset{\bullet\circ(\rho\otimes\mathrm{id}_{\mathrm{id}})}{\searrow} & & \underset{\bullet\circ(\mathrm{id}_{\mathrm{id}}\otimes\lambda)}{\swarrow} \\
 & \bullet. &
\end{array}$$

Observe that there are now two kinds of data named *associator*; the 2-cell α of C and the natural isomorphism $\alpha\colon (- \otimes -) \otimes - \Rightarrow - \otimes (- \otimes -)$ in $\mathcal{V}$. To differentiate them, we will refer to the latter as the *$\mathcal{V}$-associator*. The same applies to the other expressions that could refer to data from either C or $\mathcal{V}$.
Also note that we have omitted the $\mathcal{V}$-associator and $\mathcal{V}$-unitors from the notation; e.g. the 1-cell $\mathrm{id} \otimes \mathrm{id}^C \otimes \mathrm{id}$ is really the composition

$$C(Y,Z) \otimes C(X,Y) \cong C(Y,Z) \otimes \mathbb{1} \otimes C(X,Y) \xrightarrow{\mathrm{id}_{C(Y,Z)}\otimes\mathrm{id}^C_Y\otimes\mathrm{id}_{C(X,Y)}} C(Y,Z) \otimes C(Y,Y) \otimes C(X,Y).$$

Example 2.3.5.

1. Our motivating example occurs for the cartesian monoidal 2-category $(\mathcal{Cat}, \times, \{*\})$. By Remark 2.2.5, the $\mathcal{Cat}$-enriched categories are precisely the ordinary bicategories.

2. Viewing a monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$ as a monoidal 2-category with only identity 2-cells, the associators and unitors of a $\mathcal{V}$-enriched category must be the identity and the pentagon and triangle identities are trivially satisfied. Consequently, a $\mathcal{V}$-enriched category when $\mathcal{V}$ is regarded as a trivial 2-category is precisely the same as an ordinary $\mathcal{V}$-enriched category when $\mathcal{V}$ is viewed as a monoidal category.

Remark 2.3.6. Because all 2-cells occurring in the definition of a $\mathcal{V}$-enriched category are isomorphisms, the data of a $\mathcal{V}$-enriched category only refers to the underlying (2,1)-category of $\mathcal{V}$. ○

For ordinary monoidal categories, the functors that allow changing the enrichment (Construction 2.1.1) are the lax monoidal ones. Accordingly, the *lax monoidal functor of 2-categories* play the same role for categories enriched over monoidal 2-categories, as we will see in Construction 2.4.1. In order to stress their importance for this purpose, we will call them *changers* instead.

Definition 2.3.7. Let $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$ be two monoidal 2-categories. A **changer** is a lax monoidal functor of 2-categories; that is, a 2-functor $\mathcal{F}\colon \mathcal{V} \to \mathcal{W}$ together with a 2-natural transformation

$$\mathcal{V} \times \mathcal{V} \underset{\mathcal{F}(-\otimes-)}{\overset{\mathcal{F}(-)\otimes\mathcal{F}(-)}{\Downarrow \epsilon}} \mathcal{W}$$

and a 1-cell $u\colon \mathbb{1} \to \mathcal{F}(\mathbb{1})$ in $\mathcal{W}$, such that $(\mathcal{F}, \epsilon, u)$ constitutes a lax monoidal functor on the underlying monoidal categories.

We will always denote changers by letters like $\mathcal{F}$ or $\mathcal{G}$.

Remark 2.3.8. Explicitly, the assumption that $\epsilon\colon \mathcal{F}(-\otimes -) \Rightarrow \mathcal{F}\otimes\mathcal{F}$ is a 2-natural transformation (i.e. a $\mathcal{C}at$-enriched natural transformation), means that ϵ is not only natural in 1-cells $f\colon X \to Y$, $g\colon X' \to Y'$ in the sense that the diagram

$$\begin{array}{ccc} \mathcal{F}(X)\otimes\mathcal{F}(X') & \xrightarrow{\epsilon} & \mathcal{F}(X\otimes X') \\ {\scriptstyle \mathcal{F}(f)\otimes\mathcal{F}(g)}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(f\otimes g)} \\ \mathcal{F}(Y)\otimes\mathcal{F}(Y') & \xrightarrow{\epsilon} & \mathcal{F}(Y\otimes Y') \end{array}$$

commutes, but also in 2-cells

$$X \underset{f'}{\overset{f}{\underset{\Downarrow \alpha}{\rightrightarrows}}} Y \qquad\qquad X' \underset{g'}{\overset{g}{\underset{\Downarrow \beta}{\rightrightarrows}}} Y';$$

that is, the following two 2-cells are equal:

$$\mathcal{F}(X)\otimes\mathcal{F}(X') \xrightarrow{\epsilon} \mathcal{F}(X\otimes X') \underset{\mathcal{F}(f'\otimes g')}{\overset{\mathcal{F}(f\otimes g)}{\underset{\Downarrow \mathcal{F}(\alpha\otimes\beta)}{\rightrightarrows}}} \mathcal{F}(Y\otimes Y')$$

$$\mathcal{F}(X)\otimes\mathcal{F}(X') \underset{\mathcal{F}(f')\otimes\mathcal{F}(g')}{\overset{\mathcal{F}(f)\otimes\mathcal{F}(g)}{\underset{\Downarrow \mathcal{F}(\alpha)\otimes\mathcal{F}(\beta)}{\rightrightarrows}}} \mathcal{F}(Y)\otimes\mathcal{F}(Y') \xrightarrow{\epsilon} \mathcal{F}(Y\otimes Y').$$

This additional compatibility with the 2-cells distinguishes 2-natural transformations from the ordinary ones. ○

In practice, many functors occurring between cartesian monoidal 2-categories preserve finite products in the following sense.

Definition 2.3.9. Let $F\colon \mathcal{V} \to \mathcal{W}$ be a 2-functor between two cartesian monoidal 2-categories $(\mathcal{V}, \times, \{*\})$ and $(\mathcal{W}, \times, \{*\})$. We say that F **preserves finite products**, if we have $F(\{*\}) \cong \{*\}$ and the canonical 2-natural transformation

$$F(-\times -) \Rightarrow F(-)\times F(-)$$

induced by the universal property of the product is an isomorphism.

In words, this means that F preserves finite products of 0-cells, 1-cells and 2-cells and this is of course a stronger statement than just asking for the underlying ordinary functor to preserve finite products.

Recalling that any strong monoidal functor is in particular lax monoidal, it follows that 2-functors preserving finite products are changers in a canonical way.

Definition 2.3.10. Let $F\colon \mathcal{V} \to \mathcal{W}$ be a 2-functor between two cartesian monoidal 2-categories $(\mathcal{V}, \times, \{*\})$ and $(\mathcal{W}, \times, \{*\})$ which preserves finite products. Then $\mathcal{F} = F$ obtains the structure of a changer $(\mathcal{F}, \epsilon, u)$ with ϵ given by the inverse of the canonical 2-natural transformation

$$\mathcal{F}(-\times -) \Rightarrow \mathcal{F}(-)\times\mathcal{F}(-)$$

and u the inverse of the unique morphism $\mathcal{F}(\{*\}) \to \{*\}$. We call a changer induced in this way **cartesian**.

The famous *coherence theorem for monoidal categories* by Mac Lane (see [Mac63]) states that any "formal" diagram in a monoidal category consisting of associators and unitors commutes. The analogous statement for lax symmetric monoidal functors has been proven by Epstein (see [Eps66]) in direct reaction to Mac Lane's result.

Theorem 2.3.11 (Coherence of lax monoidal functors). Let $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$ be two monoidal categories with a lax monoidal functor

$$F\colon \mathcal{V} \to \mathcal{W}, \quad \epsilon\colon F(-) \otimes F(-) \Rightarrow F(- \otimes -), \quad u\colon \mathbb{1} \to F(\mathbb{1}).$$

Then for every $n \in \mathbb{N}$, all formal compositions of ϵ, identities and $\otimes$ of the form

$$\bigotimes_{i=1}^{n} F(-) \Rightarrow F\left(\bigotimes_{i=1}^{n} -\right)$$

are equal. We denote this composition by $\epsilon^{(n)}$ (in particular $\epsilon^{(1)} = \mathrm{id}$ and $\epsilon^{(2)} = \epsilon$).

The following remark shows that if we are only interested in cartesian monoidal 2-categories and cartesian changers between them, then the coherence of the underlying strong monoidal functors is more apparent.

Remark 2.3.12. Suppose that $(\mathcal{V}, \times, \{*\})$ and $(\mathcal{W}, \times, \{*\})$ are two cartesian monoidal categories. If $F\colon \mathcal{V} \to \mathcal{W}$ preserves finite products and ϵ is given by the inverse of the canonical natural transformation $v\colon F(- \times -) \Rightarrow F(-) \times F(-)$, then the coherence result is easy to prove.
Indeed, in this case it is equivalent to showing that there is only one formal composition

$$v^{(n)}\colon F\left(\prod_{i=1}^{n} \mathcal{E}_i\right) \Rightarrow \prod_{i=1}^{n} F(\mathcal{E}_i)$$

of v, identities and products thereof. To establish the latter, it is in turn sufficient to show the identity $\pi_i \circ v^{(n)} = F(\pi_i)$ for all $i \in \{1, \dots, n\}$.
For $n = 1$, we have $v^{(1)} = \mathrm{id}$, so we may assume that $n > 1$. Then we can write $v^{(n)}$ as the composition

$$F\left(\prod_{i=1}^{n} -\right) \xRightarrow{\ v\ } F\left(\prod_{i=1}^{k} -\right) \times F\left(\prod_{i=k+1}^{n} -\right) \xRightarrow{v^{(k)} \times v^{(n-k)}} \prod_{i=1}^{k} F(-) \times \prod_{i=k+1}^{n} F(-)$$

for some natural number $k \in \{1, \dots, n-1\}$. For $j \in \{1, \dots, k\}$ (the case $j \in \{k+1, \dots, n\}$ follows analogously), we then obtain the commutative diagram

$$\begin{array}{ccccc}
F\left(\prod_{i=1}^{n} -\right) & \xRightarrow{\ v\ } & F\left(\prod_{i=1}^{k} -\right) \times F\left(\prod_{i=k+1}^{n} -\right) & \xRightarrow{v^{(k)} \times v^{(n-k)}} & \prod_{i=1}^{k} F(-) \times \prod_{i=k+1}^{n} F(-) \\
& \searrow^{F(\pi_1)} & \Downarrow \pi_1 & & \Downarrow \pi_1 \\
& & F\left(\prod_{i=1}^{k} -\right) & \xRightarrow{\ v^{(k)}\ } & \prod_{i=1}^{k} F(-) \\
& & & \searrow^{F(\pi_j)} & \Downarrow \pi_j \\
& & & & F(-),
\end{array}$$

where the lower triangle commutes by induction. ○

2.4 Change of Enrichment for Categories enriched over monoidal 2-categories

We are finally ready to formulate and prove our change of enrichment result for categories enriched over monoidal 2-categories, generalizing Construction 2.1.1.

Construction 2.4.1 (Change of Enrichment over monoidal 2-Categories). Let $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$ be two monoidal 2-categories and $(\mathcal{F}\colon \mathcal{V} \to \mathcal{W}, \epsilon, u)$ be a changer (in the sense of Definition 2.3.7). From any $\mathcal{V}$-enriched category $(C, \bullet, \lambda, \rho, \alpha)$ we may construct a $\mathcal{W}$-enriched category $C_{\mathcal{F}} = \mathrm{ChEn}(C, \mathcal{F})$ as follows:

1. The objects of $C_{\mathcal{F}}$ are those of C; i.e. $\mathrm{Ob}(C_{\mathcal{F}}) = \mathrm{Ob}(C)$.
2. For $X, Y \in \mathrm{Ob}(C)$, we have $C_{\mathcal{F}}(X, Y) \coloneqq \mathcal{F}(C(X, Y)) \in \mathcal{W}$.
3. The identity 1-cell $\mathrm{id}_X^{C_{\mathcal{F}}}$ of $X \in \mathrm{Ob}(C)$ is
$$\mathbb{1} \xrightarrow{\;u\;} \mathcal{F}(\mathbb{1}) \xrightarrow{\mathcal{F}(\mathrm{id}_X^C)} \mathcal{F}(C(X,X)).$$
4. The horizontal composition in $C_{\mathcal{F}}$ is the 1-cell
$$\hat{\bullet}\colon\; \mathcal{F}(C(Y,Z)) \otimes \mathcal{F}(C(X,Y)) \xrightarrow{\;\epsilon\;} \mathcal{F}(C(Y,Z) \otimes C(X,Y)) \xrightarrow{\mathcal{F}(\bullet)} \mathcal{F}(C(X,Z)).$$
5. The left and right unitors of $C_{\mathcal{F}}$ are induced by those of C:
$$\hat{\lambda} \coloneqq \mathcal{F}(\lambda), \qquad \hat{\rho} \coloneqq \mathcal{F}(\rho).$$
6. With the natural transformation $\epsilon^{(3)}\colon \mathcal{F}(-)\otimes\mathcal{F}(-)\otimes\mathcal{F}(-) \Rightarrow \mathcal{F}(- \otimes - \otimes -)$ from Theorem 2.3.11, the associator $\hat{\alpha}$ in $C_{\mathcal{F}}$ is (writing $[X, Y] \coloneqq C(X, Y)$ for brevity)
$$\mathcal{F}([Y,Z]) \otimes \mathcal{F}([X,Y]) \otimes \mathcal{F}([W,X]) \xrightarrow{\epsilon^{(3)}} \mathcal{F}([Y,Z] \otimes [X,Y] \otimes [W,X]) \underset{\mathcal{F}(-\bullet(-\bullet-))}{\overset{\mathcal{F}((-\bullet-)\bullet-)}{\Downarrow \mathcal{F}(\alpha)}} \mathcal{F}([W,Z]).$$

Example 2.4.2. Our central motivation is the case of the cartesian monoidal category
$$(\mathcal{V}, \otimes, \mathbb{1}) = (\mathcal{W}, \otimes, \mathbb{1}) = (\mathcal{C}at, \times, \{*\}).$$
In this setting, changing the enrichment (in the sense of Construction 2.4.1) allows us to construct a bicategory $C_{\mathcal{F}}$ from a given bicategory C and a changer $\mathcal{F}\colon \mathcal{C}at \to \mathcal{C}at$.

We can also replace $\mathcal{V} = \mathcal{C}at$ by a monoidal 2-subcategory $\mathcal{B} \subset \mathcal{C}at$. This is necessary if the changer $\mathcal{F}$ can only be defined on a proper subcategory of C, as is for instance the case for $\mathcal{S}pan_1$ of Definition 4.1.4. However, in this case we have to ensure that the "input category" C can be regarded as a $\mathcal{B}$-enriched category; i.e. "all the data" of C must lie in $\mathcal{B}$. Explicitly, this means that C must satisfy (for all $X, Y \in \mathrm{Ob}(C)$)
$$C(X,Y) \in \mathcal{B}, \qquad \mathrm{id}_X^C \in \mathcal{B}, \qquad \bullet \in \mathcal{B}, \qquad \lambda \in \mathcal{B}, \qquad \rho \in \mathcal{B}, \qquad \alpha \in \mathcal{B}.$$
In either case, the procedure can roughly be summarized by saying that we apply $\mathcal{F}$ to all hom-categories. Observe that the identity 1-cell $\mathrm{id}_X^{C_{\mathcal{F}}}$ of $X \in \mathrm{Ob}(C)$ corresponds to the unique object in the image of the functor
$$\{*\} \xrightarrow{\;u\;} \mathcal{F}(\{*\}) \xrightarrow{\mathcal{F}(\mathrm{const}(\mathrm{id}_X))} \mathcal{F}(C(X,X)).$$

Example 2.4.3. Viewing a monoidal category $(\mathcal{D}, \otimes, \mathbb{1})$ as a bicategory with a single object via delooping (see Example 2.2.2), Example 2.4.2 shows that change of enrichment can be used to produce new monoidal categories from old ones. For example, the changer
$$\mathcal{C}at(C, -)\colon \mathcal{C}at \to \mathcal{C}at,\; \mathcal{D} \mapsto [C, \mathcal{D}]$$

from Example 3.3.1 reveals that for any category C, the functor category $[C, \mathcal{D}]$ is also a monoidal category. By Construction 2.4.1, the monoidal product on $[C, \mathcal{D}]$ is defined pointwise

$$[C, \mathcal{D}] \times [C, \mathcal{D}] \xrightarrow{\cong} [C, \mathcal{D} \times \mathcal{D}] \xrightarrow{\otimes \circ -} [C, \mathcal{D}]$$

and the unit object is the constant functor $\mathrm{const}(\mathbb{1}) \colon C \to \mathcal{D}$.
From this description and the pointwise nature of limits in functor categories, it follows that the induced monoidal structure on $[C, \mathcal{D}]$ is cartesian if $(\mathcal{D}, \otimes, \mathbb{1})$ is cartesian monoidal.

We highlight some further examples of this construction in Section 3.2, Section 3.3 and Section 4.3.
Our next goal is to prove that Construction 2.4.1 yields a well-defined $\mathcal{W}$-enriched category $C_{\mathcal{F}}$. In order to achieve this, we require a couple of lemmas.
We begin with the following lemma, which may be thought of as extending the naturality of ϵ. Indeed, if $n = 2$ and $l_1 = l_2 = 1$, then it reduces to the statement that ϵ is natural.

Lemma 2.4.4. Let $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$ be two monoidal categories and $(F \colon \mathcal{V} \to \mathcal{W}, \epsilon, u)$ be a lax monoidal functor. Furthermore, let $f_i \colon \bigotimes_{j=1}^{l_i} A_{i,j} \to B_i$ be morphisms in $\mathcal{V}$ for $i \in \{1, \dots, n\}$ and $n, l_i \in \mathbb{N}_{>0}$. Then we have

$$\epsilon^{(n)} \circ \bigotimes_{i=1}^{n} \Big(F(f_i) \circ \epsilon^{(l_i)} \Big) = F\left(\bigotimes_{i=1}^{n} f_i \right) \circ \epsilon^{(\sum_{j=1}^{n} l_j)}.$$

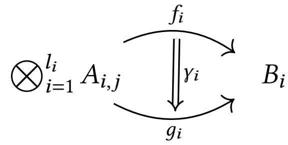
Proof. Because $\epsilon^{(1)}$ is the identity, the statement is clear for $n = 1$. The case $n > 1$ follows by Theorem 2.3.11, the naturality of ϵ and induction:

$$\begin{aligned}
F\left(\bigotimes_{i=1}^{n} f_i \right) \circ \epsilon^{(\sum_{j=1}^{n} l_j)} &= F\left(f_1 \otimes \bigotimes_{i=2}^{n} f_i \right) \circ \epsilon \circ \left(\epsilon^{(l_1)} \otimes \epsilon^{(\sum_{j=2}^{n} l_j)} \right) \\
&= \epsilon \circ \left(F(f_1) \otimes F\left(\bigotimes_{i=2}^{n} f_i \right) \right) \circ \left(\epsilon^{(l_1)} \otimes \epsilon^{(\sum_{j=2}^{n} l_j)} \right) \\
&= \epsilon \circ \left(\Big(F(f_1) \circ \epsilon^{(l_1)} \Big) \otimes \left(\epsilon^{(n-1)} \circ \bigotimes_{i=2}^{n} \Big(F(f_i) \circ \epsilon^{(l_i)} \Big) \right) \right) \\
&= \epsilon \circ (\mathrm{id} \otimes \epsilon^{(n-1)}) \circ \bigotimes_{i=1}^{n} \Big(F(f_i) \circ \epsilon^{(l_i)} \Big).
\end{aligned}$$

□

We also require this extended form of naturality for the 2-cells in a monoidal 2-category, which follows with exactly the same proof as the previous lemma. Note that this uses our assumption that α is 2-natural, so that a 2-cell like $(\alpha \otimes \beta) \otimes \gamma$ can be identified with $\alpha \otimes (\beta \otimes \gamma)$.

Lemma 2.4.5. Let $\mathcal{F} \colon \mathcal{V} \to \mathcal{W}$ be a changer between two monoidal 2-categories $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$. Then 2-cells



for $i \in \{1, \dots, n\}$, $l_i \in \mathbb{N}_{>0}$ satisfy

$$\epsilon^{(n)} \circ \bigotimes_{i=1}^{n} \Big(\mathcal{F}(\gamma_i) \circ \epsilon^{(l_i)} \Big) = \mathcal{F}\left(\bigotimes_{i=1}^{n} \gamma_i \right) \circ \epsilon^{(\sum_{j=1}^{n} l_j)}.$$

The final lemma describes the compatibility of a lax monoidal functor with the unitors. Note that the isomorphisms I and J from the lemma are usually suppressed from the notation, except in the statement and proof of the lemma.

Lemma 2.4.6. Let $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$ be two monoidal categories and $n \in \mathbb{N}$, $n \geq 2$. For any lax monoidal functor $(F\colon \mathcal{V} \to \mathcal{W}, \epsilon, u)$ and morphisms $f_i\colon A_i \to B_i$ in $\mathcal{V}$ ($i \in \{1, \dots, n\}$) such that $A_j = \mathbb{1}$ for some $j \in \{1, \dots, n\}$, we have

$$F\left(\bigotimes_{i=1}^{n} f_i\right) \circ F(I) \circ \epsilon^{(n-1)} = \epsilon^{(n)} \circ \left(\bigotimes_{i=1}^{j-1} F(f_i) \otimes \big(F(f_j) \circ u\big) \otimes \bigotimes_{i=j+1}^{n} F(f_i)\right) \circ J,$$

where $I\colon \bigotimes_{i \in \{1,\dots,n\}\setminus\{j\}} A_i \to \bigotimes_{i=1}^{n} A_i$ is the unique composition of $\otimes$, $\mathcal{V}$-unitors and identities and similarly for $J\colon \bigotimes_{i \in \{1,\dots,n\}\setminus\{j\}} F(A_i) \to \bigotimes_{i=1}^{j-1} F(A_i) \otimes \mathbb{1} \otimes \bigotimes_{i=j+1}^{n} F(A_i)$.
For the case $n = 2$, this means that any two morphisms $f\colon A \to A'$ and $g\colon \mathbb{1} \to B$ in $\mathcal{V}$ satisfy

$$F(f \otimes g) \circ F(I) = \epsilon \circ (F(f) \otimes (F(g) \circ u)) \circ J, \qquad F(g \otimes f) \circ F(I) = \epsilon \circ ((F(g) \circ u) \otimes F(f)) \circ J.$$

Proof. By Lemma 2.4.4, the right square in the diagram

$$\begin{array}{ccccccc}
\bigotimes_{i \in \{1,\dots,n\}\setminus\{j\}} F(A_i) & \xrightarrow{\epsilon^{(n-1)}} & F\Big(\bigotimes_{i \in \{1,\dots,n\}\setminus\{j\}} A_i\Big) & \xrightarrow{F(I)} & F\big(\bigotimes_{i=1}^{n} A_i\big) & \xrightarrow{F(\bigotimes_{i=1}^{n} f_i)} & F\big(\bigotimes_{i=1}^{n} B_i\big) \\
& \searrow^{J} & & & \uparrow{\scriptstyle \epsilon^{(n)}} & & \uparrow{\scriptstyle \epsilon^{(n)}} \\
& & \bigotimes_{i=1}^{j-1} F(A_i) \otimes \mathbb{1} \otimes \bigotimes_{i=j+1}^{n} F(A_i) & \xrightarrow{\mathrm{id} \otimes u \otimes \mathrm{id}} & \bigotimes_{i=1}^{n} F(A_i) & \xrightarrow{\bigotimes_{i=1}^{n} F(f_i)} & \bigotimes_{i=1}^{n} F(B_i)
\end{array}$$

commutes, so it remains to verify that the rest also commutes.
Assuming $j > 1$, we may write $I = \mathrm{id} \otimes \rho^{-1} \otimes \mathrm{id}$ with $\rho\colon A_{j-1} \otimes \mathbb{1} \to A_{j-1}$ denoting the right $\mathcal{V}$-unitor. Therefore, utilizing the lemma again yields

$$F(I) \circ \epsilon^{(n-1)} = F\big(\mathrm{id} \otimes \rho^{-1} \otimes \mathrm{id}\big) \circ \epsilon^{(n-1)} = \epsilon^{(n-1)} \circ \big(\mathrm{id} \otimes F(\rho^{-1}) \otimes \mathrm{id}\big).$$

Similarly, with $\tilde{\rho}\colon F(A_{j-1}) \otimes \mathbb{1} \to F(A_{j-1})$ denoting the right $\mathcal{W}$-unitor, we have $J = \mathrm{id} \otimes \tilde{\rho}^{-1} \otimes \mathrm{id}$ and thus by the unitality of F

$$\begin{aligned}
\epsilon^{(n)} \circ (\mathrm{id} \otimes u \otimes \mathrm{id}) \circ J &= \epsilon^{(n-1)} \circ (\mathrm{id} \otimes \epsilon \otimes \mathrm{id}) \circ (\mathrm{id} \otimes u \otimes \mathrm{id}) \circ \big(\mathrm{id} \otimes \tilde{\rho}^{-1} \otimes \mathrm{id}\big) \\
&= \epsilon^{(n-1)} \circ \big(\mathrm{id} \otimes \big(\epsilon \circ (\mathrm{id} \otimes u) \circ \tilde{\rho}^{-1}\big) \otimes \mathrm{id}\big) = \epsilon^{(n-1)} \circ \big(\mathrm{id} \otimes F(\rho^{-1}) \otimes \mathrm{id}\big).
\end{aligned}$$

This shows the desired equality if $j > 1$. The case $j = 1$ follows similarly, using the left unitors instead. □

We can now proceed to prove that our construction indeed produces a valid $\mathcal{W}$-enriched category.

Theorem 2.4.7. Given a changer $(\mathcal{F}\colon \mathcal{V} \to \mathcal{W}, \epsilon, u)$ between two monoidal 2-categories $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$ and a $\mathcal{V}$-enriched category C, Construction 2.4.1 yields a well-defined $\mathcal{W}$-enriched category $C_{\mathcal{F}}$.

Proof. First we observe that the domain (for the codomain this is immediate) of the right unitor

$$\hat{\rho} = \mathcal{F}(\rho)\colon \mathcal{F}(- \bullet \mathrm{id}_X^C) \overset{\cong}{\Longrightarrow} \mathcal{F}(\mathrm{id})$$

is correct by Lemma 2.4.6:

$$\begin{aligned}
\mathcal{F}(- \bullet \mathrm{id}^C) &= \mathcal{F}\Big(\bullet \circ \Big(\mathrm{id} \otimes \mathrm{id}^C\Big)\Big) = \mathcal{F}(\bullet) \circ \mathcal{F}\Big(\mathrm{id} \otimes \mathrm{id}^C\Big) \\
&= \mathcal{F}(\bullet) \circ \epsilon \circ \mathcal{F}(\mathrm{id}) \otimes \Big(\mathcal{F}(\mathrm{id}^C) \circ u\Big) = \hat{\bullet} \circ (\mathrm{id} \otimes \mathrm{id}^{C_{\mathcal{F}}}) = -\hat{\bullet}\mathrm{id}^{C_{\mathcal{F}}}.
\end{aligned}$$

An analogous argument shows the same for the left unitor $\hat{\lambda}$. The domain of the associator $\hat{\alpha}$ is also correct, which we verify with the calculation (utilizing Lemma 2.4.4)

$$\begin{aligned}
\mathcal{F}((- \bullet -) \bullet -) \circ \epsilon^{(3)} &= \mathcal{F}(\bullet \circ (\bullet \otimes \mathrm{id})) \circ \epsilon^{(3)} = \mathcal{F}(\bullet) \circ \mathcal{F}(\bullet \otimes \mathrm{id}) \circ \epsilon^{(3)} \\
&= \mathcal{F}(\bullet) \circ \epsilon \circ (\mathcal{F}(\bullet) \circ \epsilon) \otimes \mathrm{id} = (-\hat{\bullet}-)\hat{\bullet}-
\end{aligned}$$

and the same applies to the codomain.
It remains to show that the pentagon and triangle identities are satisfied. To see this, we start with the triangle identity in C

$$\begin{array}{ccc} \bullet \circ (\bullet \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id}^C \otimes \mathrm{id}) & \xRightarrow{\alpha \circ (\mathrm{id} \otimes \mathrm{id}^C \otimes \mathrm{id})} & \bullet \circ (\mathrm{id} \otimes \bullet) \circ (\mathrm{id} \otimes \mathrm{id}^C \otimes \mathrm{id}) \\ & {\scriptstyle \bullet \circ (\rho \otimes \mathrm{id}_{\mathrm{id}})} \searrow \quad \swarrow {\scriptstyle \bullet \circ (\mathrm{id}_{\mathrm{id}} \otimes \lambda)} & \\ & \bullet. & \end{array}$$

Applying $\mathcal{F}$ to this diagram and precomposing by ϵ yields the following 2-cells in $\mathcal{W}$:

$$\begin{aligned} \mathcal{F}\Big(\alpha \circ (\mathrm{id} \otimes \mathrm{id}^C \otimes \mathrm{id})\Big) \circ \epsilon &= \mathcal{F}(\alpha) \circ \mathcal{F}\Big(\mathrm{id} \otimes \mathrm{id}^C \otimes \mathrm{id}\Big) \circ \epsilon = \mathcal{F}(\alpha) \circ \epsilon^{(3)} \circ \Big(\mathrm{id} \otimes \Big(\mathcal{F}(\mathrm{id}^C) \circ u\Big) \otimes \mathrm{id}\Big) \\ &= \hat{\alpha} \circ (\mathrm{id} \otimes \mathrm{id}^{C_{\mathcal{F}}} \otimes \mathrm{id}), \\ \mathcal{F}(\bullet \circ (\rho \otimes \mathrm{id}_{\mathrm{id}})) \circ \epsilon &= \mathcal{F}(\bullet) \circ \epsilon \circ (\mathcal{F}(\rho) \otimes \mathrm{id}_{\mathrm{id}}) = \hat{\bullet} \circ (\hat{\rho} \otimes \mathrm{id}_{\mathrm{id}}), \\ \mathcal{F}(\bullet \circ (\mathrm{id}_{\mathrm{id}} \otimes \lambda)) \circ \epsilon &= \mathcal{F}(\bullet) \circ \epsilon \circ (\mathrm{id}_{\mathrm{id}} \otimes \mathcal{F}(\lambda)) = \hat{\bullet} \circ \Big(\mathrm{id}_{\mathrm{id}} \otimes \hat{\lambda}\Big). \end{aligned}$$

Here we used Lemma 2.4.6 for the first equation and the 2-naturality of ϵ (see Remark 2.3.8) for the other two. The calculation shows that the corresponding diagram constitutes the desired triangle identity for $C_{\mathcal{F}}$.
Similarly, the pentagon identity of $C_{\mathcal{F}}$ follows from the one in C by applying $\mathcal{F}$ and precomposing with $\epsilon^{(4)}$, because by Lemma 2.4.4 and Lemma 2.4.5:

$$\begin{aligned} \mathcal{F}(\alpha \circ (\mathrm{id} \otimes \mathrm{id} \otimes \bullet)) \circ \epsilon^{(4)} &= \mathcal{F}(\alpha) \circ \epsilon^{(3)} \circ (\mathrm{id} \otimes \mathrm{id} \otimes (\mathcal{F}(\bullet) \circ \epsilon)) = \hat{\alpha} \circ (\mathrm{id} \otimes \mathrm{id} \otimes \hat{\bullet}), \\ \mathcal{F}(\bullet \circ (\alpha \otimes \mathrm{id}_{\mathrm{id}})) \circ \epsilon^{(4)} &= \mathcal{F}(\bullet) \circ \epsilon \circ \Big(\Big(\mathcal{F}(\alpha) \circ \epsilon^{(3)}\Big) \otimes \mathrm{id}_{\mathrm{id}}\Big) = \hat{\bullet} \circ (\hat{\alpha} \otimes \mathrm{id}_{\mathrm{id}}). \end{aligned}$$

The remaining equations follow analogously. □

Remark 2.4.8. Viewing monoidal categories as monoidal 2-categories with only identity 2-cells, a changer between two such monoidal categories is just a lax monoidal functor. Change of enrichment using Construction 2.4.1 produces the same monoidal category as Construction 2.1.1.
In this sense, the former construction generalizes the latter. ○

2.5 Functoriality of Change of Enrichment

Recalling the functoriality result for categories enriched over ordinary monoidal categories (Theorem 2.1.4), it seems quite natural that a similar statement should apply when changing the enrichment of categories enriched over monoidal 2-categories. Indeed, Theorem 2.6.2 will exhibit ChEn as the action on hom-categories of a strict functor

$$2\mathcal{C}at_{\otimes}^{lax} \to \mathcal{C}at, \ \mathcal{V} \mapsto \mathcal{V}\text{-}\mathcal{C}at.$$

As a first step, we want to realize the functoriality of $\mathrm{ChEn}(C, \mathcal{F})$ in the "input category" C. In order to make this precise, we require a notion of morphisms between $\mathcal{V}$-enriched categories, corresponding (in case they are enriched over $(\mathcal{C}at, \times, \{*\})$) to the different kinds of functors between bicategories.

Definition 2.5.1. Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal 2-category and C and $\mathcal{D}$ be two $\mathcal{V}$-enriched categories. A **lax $\mathcal{V}$-enriched functor** $F\colon C \to \mathcal{D}$ consists of the following data:

1. A function $F\colon \mathrm{Ob}(C) \to \mathrm{Ob}(\mathcal{D})$ mapping objects to objects.

2. For each pair of objects $X, Y \in \mathrm{Ob}(C)$, a 1-cell in $\mathcal{V}$

$$F = F(X, Y)\colon C(X, Y) \to \mathcal{D}(F(X), F(Y)).$$

3. For every object $X \in \mathrm{Ob}(\mathcal{C})$, a 2-cell

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\quad \mathrm{id}^{\mathcal{D}}_{F(X)} \quad} & \mathcal{D}(F(X),F(X)) \\ & \mathrm{id}^{\mathcal{C}}_X \searrow \quad \Downarrow F_{\mathrm{id}} \quad \nearrow F(X,X) & \\ & \mathcal{C}(X,X), & \end{array}$$

called *unity constraint*.

4. For all objects $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a 2-cell

$$\begin{array}{ccc} \mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y) & \xrightarrow{F \otimes F} & \mathcal{D}(F(Y),F(Z)) \otimes \mathcal{D}(F(X),F(Y)) \\ \downarrow \bullet & \swArrow F_\bullet & \bullet \downarrow \\ \mathcal{C}(X,Z) & \xrightarrow{\quad F \quad} & \mathcal{D}(F(X),F(Z)), \end{array}$$

called *functoriality constraint*.

This data is subject to the following commutative diagrams in $\mathcal{V}$:

1. *Associativity*:

$$\begin{array}{ccc} \bullet \circ (\bullet \otimes \mathrm{id}) \circ (F \otimes F \otimes F) & \xRightarrow{\alpha \circ (F \otimes F \otimes F)} & \bullet \circ (\mathrm{id} \otimes \bullet) \circ (F \otimes F \otimes F) \\ {\scriptstyle \bullet \circ (F_\bullet \otimes \mathrm{id}_F)} \Downarrow & & \Downarrow {\scriptstyle \bullet \circ (\mathrm{id}_F \otimes F_\bullet)} \\ \bullet \circ ((F \circ \bullet) \otimes F) & & \bullet \circ (F \otimes (F \circ \bullet)) \\ {\scriptstyle F_\bullet \circ (\bullet \otimes \mathrm{id})} \Downarrow & & \Downarrow {\scriptstyle F_\bullet \circ (\mathrm{id} \otimes \bullet)} \\ F \circ \bullet \circ (\bullet \otimes \mathrm{id}) & \xRightarrow{\quad F \circ \alpha \quad} & F \circ \bullet \circ (\mathrm{id} \otimes \bullet). \end{array}$$

2. *Unity*:

$$\begin{array}{ccc} \bullet \circ (\mathrm{id}^{\mathcal{D}} \otimes F) & \xRightarrow{\quad \lambda \circ F \quad} & F \\ \Downarrow {\scriptstyle \bullet \circ (F_{\mathrm{id}} \otimes \mathrm{id}_F)} & & \Uparrow {\scriptstyle F \circ \lambda} \\ \bullet \circ ((F \circ \mathrm{id}^{\mathcal{C}}) \otimes F) & \xRightarrow{F_\bullet \circ (\mathrm{id}^{\mathcal{C}} \otimes \mathrm{id})} & F \circ \bullet \circ (\mathrm{id}^{\mathcal{C}} \otimes \mathrm{id}) \end{array}$$

$$\begin{array}{ccc} \bullet \circ (F \otimes \mathrm{id}^{\mathcal{D}}) & \xRightarrow{\quad \rho \circ F \quad} & F \\ \Downarrow {\scriptstyle \bullet \circ (\mathrm{id}_F \otimes F_{\mathrm{id}})} & & \Uparrow {\scriptstyle F \circ \rho} \\ \bullet \circ (F \otimes (F \circ \mathrm{id}^{\mathcal{C}})) & \xRightarrow{F_\bullet \circ (\mathrm{id} \otimes \mathrm{id}^{\mathcal{C}})} & F \circ \bullet \circ (\mathrm{id} \otimes \mathrm{id}^{\mathcal{C}}). \end{array}$$

Here we have as usual suppressed the $\mathcal{V}$-unitors and $\mathcal{V}$-associators from the notation; e.g. $\bullet \circ (\mathrm{id}^{\mathcal{D}} \otimes F)$ is actually the 1-cell

$$\mathcal{C}(X,Y) \cong \mathbb{1} \otimes \mathcal{C}(X,Y) \xrightarrow{\mathrm{id}^{\mathcal{D}}_{F(Y)} \otimes F(X,Y)} \mathcal{D}(F(Y),F(Y)) \otimes \mathcal{D}(F(X),F(Y)) \xrightarrow{\bullet} \mathcal{D}(F(X),F(Y)).$$

Note that by naturality of the left $\mathcal{V}$-unitor, the first part of this composition may also be written as

$$\mathcal{C}(X,Y) \xrightarrow{F} \mathcal{D}(F(X),F(Y)) \cong \mathbb{1} \otimes \mathcal{D}(F(X),F(Y)) \xrightarrow{\mathrm{id}^{\mathcal{D}}_{F(Y)} \otimes \mathrm{id}} \mathcal{D}(F(Y),F(Y)) \otimes \mathcal{D}(F(X),F(Y)).$$

If the unity and functoriality constraint are isomorphisms, then we call F a **$\mathcal{V}$-enriched functor** and if they are the identity, we refer to F as a **strict $\mathcal{V}$-enriched functor**.

Example 2.5.2.

1. Of course, a $(\mathcal{C}at, \times, \{*\})$-enriched (lax) functor is precisely a (lax) functor between bicategories because of Remark 2.2.7.
2. Viewing monoidal categories as monoidal 2-categories with only identity 2-cells, a (lax) enriched functor is just an enriched functor.

Lax $\mathcal{V}$-enriched functors can be composed "in the obvious way".

Lemma 2.5.3. Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal 2-category and $F\colon \mathcal{C} \to \mathcal{D}$, $G\colon \mathcal{D} \to \mathcal{E}$ be two lax $\mathcal{V}$-enriched functors between $\mathcal{V}$-enriched categories.
The **composition** $G \circ F\colon \mathcal{C} \to \mathcal{E}$ of F and G is the following lax $\mathcal{V}$-enriched functor:

1. Its action on objects is the composition $G \circ F\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{E})$.
2. For $X, Y \in \mathrm{Ob}(\mathcal{C})$, the corresponding 1-cell in $\mathcal{V}$ is the composition
$$\mathcal{C}(X,Y) \xrightarrow{F(X,Y)} \mathcal{D}(F(X),F(Y)) \xrightarrow{G(F(X),F(Y))} \mathcal{E}(G(F(X)),G(F(Y))).$$
3. For $X \in \mathrm{Ob}(\mathcal{C})$, the unity constraint is given by the pasting diagram
$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\mathrm{id}^{\mathcal{E}}} & \mathcal{E}(G(F(X)),G(F(X))) \\ {\scriptstyle \mathrm{id}^{\mathcal{C}}}\downarrow & \overset{F_{\mathrm{id}}}{\Leftarrow}\ \mathrm{id}^{\mathcal{D}}\ \overset{G_{\mathrm{id}}}{\Leftarrow} & \uparrow{\scriptstyle G} \\ \mathcal{C}(X,X) & \xrightarrow{F} & \mathcal{D}(F(X),F(X)). \end{array} \tag{2.1}$$
4. For $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, the functoriality constraint is the 2-cell
$$\begin{array}{ccccc} \mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y) & \xrightarrow{F\otimes F} & \mathcal{D}(Y',Z')\otimes\mathcal{D}(X',Y') & \xrightarrow{G\otimes G} & \mathcal{E}(G(Y'),G(Z'))\otimes\mathcal{E}(G(X'),G(Y')) \\ {\scriptstyle\bullet}\downarrow & \overset{F_\bullet}{\Leftarrow} & {\scriptstyle\bullet}\downarrow & \overset{G_\bullet}{\Leftarrow} & {\scriptstyle\bullet}\downarrow \\ \mathcal{C}(X,Z) & \xrightarrow{F} & \mathcal{D}(X',Z') & \xrightarrow{G} & \mathcal{E}(G(X'),G(Y')), \end{array}$$
where we abbreviated the image $F(W)$ of an object $W \in \mathrm{Ob}(\mathcal{C})$ by W'.

Proof. One has to check that $G \circ F$ is associative and unital in the sense that the three diagrams of 2-cells in Definition 2.5.1 commute. We demonstrate this for one of the unity diagrams (with the commutativity of the other two diagrams following via a similar diagram chase), namely the outer rectangle in the diagram

$$\begin{array}{ccccc} \bullet\circ(\mathrm{id}^{\mathcal{E}}\otimes(G\circ F)) & & \xRightarrow{\lambda\circ G\circ F} & & G\circ F \\ \Downarrow{\scriptstyle \bullet\circ G_{\mathrm{id}}\otimes\mathrm{id}_{G\circ F}} & & & \overset{G\circ\lambda\circ F}{\nearrow} & \Uparrow{\scriptstyle G\circ F\circ\lambda} \\ \bullet\circ(G\circ\mathrm{id}^{\mathcal{D}})\otimes(G\circ F) & \xRightarrow{G_\bullet\circ\mathrm{id}^{\mathcal{D}}\otimes F} & G\circ\bullet\circ(\mathrm{id}^{\mathcal{D}}\otimes F) & & \\ \Downarrow{\scriptstyle \bullet\circ(G\circ F_{\mathrm{id}})\otimes\mathrm{id}_{G\circ F}} & & \Downarrow{\scriptstyle G\circ\bullet\circ(F_{\mathrm{id}}\otimes\mathrm{id}_F)} & & \\ \bullet\circ(G\circ F\circ\mathrm{id}^{\mathcal{C}})\otimes(G\circ F) & \xRightarrow{G_\bullet\circ(F\circ\mathrm{id}^{\mathcal{C}})\otimes F} & G\circ\bullet\circ(F\circ\mathrm{id}^{\mathcal{C}})\otimes F & \xRightarrow{G\circ F_\bullet\circ(\mathrm{id}^{\mathcal{C}}\otimes\mathrm{id})} & G\circ F\circ\bullet\circ(\mathrm{id}^{\mathcal{C}}\otimes\mathrm{id}). \end{array}$$

The top part is the unity diagram of G precomposed by F and the right part is the unity diagram of F postcomposed by G, so they both commute. Finally, the bottom left rectangle describes the two possible ways to horizontally compose the 2-cells (writing $X' \coloneqq F(X)$, $Y' \coloneqq F(Y)$)

$$\mathcal{C}(X,Y) \underset{(F\circ\mathrm{id}^{\mathcal{C}})\otimes F}{\overset{\mathrm{id}^{\mathcal{D}}\otimes F}{\rightrightarrows}} \ \Downarrow{\scriptstyle F_{\mathrm{id}}\otimes\mathrm{id}_F}\ \mathcal{D}(Y',Y')\otimes\mathcal{D}(X',Y') \underset{G\circ\bullet}{\overset{\bullet\circ G\otimes G}{\rightrightarrows}} \ \Downarrow{\scriptstyle G_\bullet}\ \mathcal{E}(G(X'),G(Y'))$$

in $\mathcal{V}$, which are equal since composition is functorial. $\square$

Of course, the point is that this allows us to define a category of $\mathcal{V}$-enriched categories.

Definition 2.5.4. For a fixed monoidal 2-category $(\mathcal{V}, \otimes, \mathbb{1})$, the category $\mathcal{V}\text{-}\mathcal{C}at$ consists of the $\mathcal{V}$-enriched categories as objects and the lax $\mathcal{V}$-enriched functors as morphisms. The composition is defined as in Lemma 2.5.3.

Example 2.5.5. Combining Example 2.3.5 and Example 2.5.2, we see that $\mathcal{C}at\text{-}\mathcal{C}at = \mathcal{B}icat$ is the category of (small) bicategories and that $(\mathcal{V}, \otimes, \mathbb{1})\text{-}\mathcal{C}at$ is the category of (small) $\mathcal{V}$-enriched categories, when viewing a monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$ as a 2-monoidal one with only identity 2-cells.

We can now show that changing the enrichment is indeed functorial.

Theorem 2.5.6 (Functoriality of Change of Enrichment I). Let $(\mathcal{F}: \mathcal{V} \to \mathcal{W}, \epsilon, u)$ be a changer between two monoidal 2-categories $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$. Construction 2.4.1 extends to a functor

$$\mathrm{ChEn}(-, \mathcal{F}) : \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{W}\text{-}\mathcal{C}at,\ C \mapsto C_{\mathcal{F}},\ G \mapsto G_{\mathcal{F}},$$

where the lax $\mathcal{W}$-enriched functor $G_{\mathcal{F}} : C_{\mathcal{F}} \to \mathcal{D}_{\mathcal{F}}$ induced by $G : C \to \mathcal{D}$ is given as follows:

1. Its action on objects $\mathrm{Ob}(C_{\mathcal{F}}) = \mathrm{Ob}(C) \to \mathrm{Ob}(\mathcal{D}) = \mathrm{Ob}(\mathcal{D}_{\mathcal{F}})$ is G.
2. For a pair of objects $X, Y \in \mathrm{Ob}(C)$, its action is given by
$$\mathcal{F}(G(X,Y)) : C_{\mathcal{F}}(X,Y) = \mathcal{F}(C(X,Y)) \to \mathcal{F}(\mathcal{D}(G(X), G(Y))) = \mathcal{D}_{\mathcal{F}}(G(X), G(Y)).$$
3. The unity constraint $(G_{\mathcal{F}})_{\mathrm{id}} : \mathrm{id}^{\mathcal{D}_{\mathcal{F}}} \Rightarrow G_{\mathcal{F}} \circ \mathrm{id}^{C_{\mathcal{F}}}$ is the 2-cell
$$\begin{array}{ccccc} \mathbb{1} & \xrightarrow{u} & \mathcal{F}(\mathbb{1}) & \xrightarrow{\mathcal{F}\left(\mathrm{id}^{\mathcal{D}}_{G(X)}\right)} & \mathcal{F}(\mathcal{D}(G(X), G(X))) \\ & & {\scriptstyle \mathcal{F}\left(\mathrm{id}^{C}_{X}\right)} \searrow & \Downarrow {\scriptstyle \mathcal{F}(G_{\mathrm{id}})} & \nearrow {\scriptstyle \mathcal{F}(G(X,X))} \\ & & & \mathcal{F}(C(X,X)). & \end{array}$$
4. The functoriality constraint $(G_{\mathcal{F}})_{\bullet} : \hat{\bullet} \circ (G_{\mathcal{F}} \otimes G_{\mathcal{F}}) \Rightarrow G_{\mathcal{F}} \circ \hat{\bullet}$ is
$$\begin{array}{ccc} C_{\mathcal{F}}(Y,Z) \otimes C_{\mathcal{F}}(X,Y) & \xrightarrow{\mathcal{F}(G) \otimes \mathcal{F}(G)} & \mathcal{D}_{\mathcal{F}}(G(Y), G(Z)) \otimes \mathcal{D}_{\mathcal{F}}(G(X), G(Y)) \\ \hat{\bullet} \downarrow & \Swarrow {\scriptstyle \mathcal{F}(G_{\bullet}) \circ \epsilon} & \downarrow \hat{\bullet} \\ C_{\mathcal{F}}(X,Z) & \xrightarrow[\mathcal{F}(G)]{} & \mathcal{D}_{\mathcal{F}}(G(X), G(Z)). \end{array}$$

Proof. First observe that the functoriality constraint is well-defined, because expanding the definition of $\hat{\bullet}$ yields the diagram

$$\begin{array}{ccc} C_{\mathcal{F}}(Y,Z) \otimes C_{\mathcal{F}}(X,Y) & \xrightarrow{\mathcal{F}(G) \otimes \mathcal{F}(G)} & \mathcal{D}_{\mathcal{F}}(G(Y), G(Z)) \otimes \mathcal{D}_{\mathcal{F}}(G(X), G(Y)) \\ \epsilon \downarrow & & \downarrow \epsilon \\ \mathcal{F}(C(Y,Z) \otimes C(X,Y)) & \xrightarrow{\mathcal{F}(G \otimes G)} & \mathcal{F}(\mathcal{D}(G(Y), G(Z)) \otimes \mathcal{D}(G(X), G(Y))) \\ \mathcal{F}(\bullet) \downarrow & \Swarrow {\scriptstyle \mathcal{F}(G_{\bullet})} & \downarrow \mathcal{F}(\bullet) \\ C_{\mathcal{F}}(X,Z) & \xrightarrow[\mathcal{F}(G)]{} & \mathcal{D}_{\mathcal{F}}(G(X), G(Z)), \end{array}$$

whose upper square commutes by naturality of ϵ.
To see that $G_{\mathcal{F}}$ is a lax $\mathcal{W}$-enriched functor, we have to verify the associativity and unity axioms. Because $G\colon \mathcal{C} \to \mathcal{D}$ is a lax $\mathcal{V}$-enriched functor, the diagram

$$\begin{array}{ccc}
\bullet \circ (\mathrm{id}^{\mathcal{D}} \otimes G) & \xRightarrow{\lambda \circ G} & G \\
\Downarrow{\scriptstyle \bullet\circ(G_{\mathrm{id}}\otimes \mathrm{id}_G)} & & \Uparrow{\scriptstyle G\circ\lambda} \\
\bullet \circ ((G \circ \mathrm{id}^{\mathcal{C}}) \otimes G) & \xRightarrow{G_\bullet \circ (\mathrm{id}^{\mathcal{C}} \otimes \mathrm{id})} & G \circ \bullet \circ (\mathrm{id}^{\mathcal{C}} \otimes \mathrm{id})
\end{array}$$

expressing its unity is commutative. By Lemma 2.4.6 and the definitions in Construction 2.4.1, we have

$$\begin{aligned}
\mathcal{F}(\bullet \circ (G_{\mathrm{id}} \otimes \mathrm{id}_G)) &= \mathcal{F}(\bullet) \circ \epsilon \circ ((\mathcal{F}(G_{\mathrm{id}}) \circ u) \otimes \mathcal{F}(\mathrm{id}_G)) = \hat{\bullet} \circ (G_{\mathcal{F}})_{\mathrm{id}} \otimes \mathrm{id}_{G_{\mathcal{F}}}, \\
\mathcal{F}\Big(G_\bullet \circ (\mathrm{id}^{\mathcal{C}} \otimes \mathrm{id})\Big) &= \mathcal{F}(G_\bullet) \circ \epsilon \circ \Big((\mathcal{F}(\mathrm{id}^{\mathcal{C}}) \circ u) \otimes \mathcal{F}(\mathrm{id})\Big) = (G_{\mathcal{F}})_\bullet \circ \mathrm{id}^{\mathcal{C}_{\mathcal{F}}} \otimes \mathrm{id}, \\
\mathcal{F}(\lambda \circ G) &= \mathcal{F}(\lambda) \circ \mathcal{F}(G) = \hat{\lambda} \circ G_{\mathcal{F}}, \\
\mathcal{F}(G \circ \lambda) &= \mathcal{F}(G) \circ \mathcal{F}(\lambda) = G_{\mathcal{F}} \circ \hat{\lambda},
\end{aligned}$$

so applying $\mathcal{F}$ to the diagram yields the corresponding unity diagram for $G_{\mathcal{F}}$. The commutativity of the other unity diagram of $G_{\mathcal{F}}$ follows analogously.
Likewise, in order to establish the associativity diagram of $G_{\mathcal{F}}$, we start with the corresponding diagram of G

$$\begin{array}{ccc}
\bullet \circ (\bullet \otimes \mathrm{id}) \circ (G \otimes G \otimes G) & \xRightarrow{\alpha \circ (G \otimes G \otimes G)} & \bullet \circ (\mathrm{id} \otimes \bullet) \circ (G \otimes G \otimes G) \\
{\scriptstyle \bullet\circ(G_\bullet \otimes \mathrm{id}_G)} \Downarrow & & \Downarrow {\scriptstyle \bullet\circ(\mathrm{id}_G \otimes G_\bullet)} \\
\bullet \circ ((G \circ \bullet) \otimes G) & & \bullet \circ (G \otimes (G \circ \bullet)) \\
{\scriptstyle G_\bullet\circ(\bullet \otimes \mathrm{id})} \Downarrow & & \Downarrow {\scriptstyle G_\bullet\circ(\mathrm{id} \otimes \bullet)} \\
G \circ \bullet \circ (\bullet \otimes \mathrm{id}) & \xRightarrow{G \circ \alpha} & G \circ \bullet \circ (\mathrm{id} \otimes \bullet).
\end{array}$$

Applying $\mathcal{F}$ to this diagram and precomposing with $\epsilon^{(3)}$ yields the desired associativity diagram for $G_{\mathcal{F}}$, which we demonstrate with the calculations (using Lemma 2.4.4 and Lemma 2.4.5)

$$\begin{aligned}
\mathcal{F}(\alpha \circ (G \otimes G \otimes G)) \circ \epsilon^{(3)} &= \mathcal{F}(\alpha) \circ \epsilon^{(3)} \circ (\mathcal{F}(G) \otimes \mathcal{F}(G) \otimes \mathcal{F}(G)) = \hat{\alpha} \circ (G_{\mathcal{F}} \otimes G_{\mathcal{F}} \otimes G_{\mathcal{F}}), \\
\mathcal{F}(\bullet \circ (G_\bullet \otimes \mathrm{id}_G)) \circ \epsilon^{(3)} &= \mathcal{F}(\bullet) \circ \epsilon \circ ((\mathcal{F}(G_\bullet) \circ \epsilon) \otimes \mathcal{F}(\mathrm{id}_G)) = \hat{\bullet} \circ ((G_{\mathcal{F}})_\bullet \otimes \mathrm{id}_{G_{\mathcal{F}}}), \\
\mathcal{F}(\bullet \circ (\mathrm{id}_G \otimes G_\bullet)) \circ \epsilon^{(3)} &= \mathcal{F}(\bullet) \circ \epsilon \circ (\mathcal{F}(\mathrm{id}_G) \otimes (\mathcal{F}(G_\bullet) \circ \epsilon)) = \hat{\bullet} \circ (\mathrm{id}_{G_{\mathcal{F}}} \otimes (G_{\mathcal{F}})_\bullet),
\end{aligned}$$

noting that the other three equations follow analogously.
It remains to check that $\mathrm{ChEn}(-, \mathcal{F})\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{W}\text{-}\mathcal{C}at$ is a functor. It is straightforward to verify that identities are preserved; that is, we have $\mathrm{ChEn}(\mathrm{id}_{\mathcal{C}}, \mathcal{F}) = \mathrm{id}_{\mathcal{C}_{\mathcal{F}}}$. To see that composition is also preserved, let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be two lax $\mathcal{V}$-enriched functors. Clearly, the action on objects and 1-cells respects composition, so it is left to check the same for the unity and functoriality constraint.
Regarding the functoriality constraint, observe that because composition is functorial and ϵ is a natural transformation, first horizontally composing $\mathcal{F}(F_\bullet)$ with $\mathcal{F}(G_\bullet)$ and then precomposing ϵ (which by definition is $\mathrm{ChEn}(G \circ F, \mathcal{F})_\bullet$) is the same as first precomposing both $\mathcal{F}(F_\bullet)$ and $\mathcal{F}(G_\bullet)$ by ϵ and then composing horizontally:

$$\begin{array}{ccccc}
\mathcal{C}_{\mathcal{F}}(Y,Z) \otimes \mathcal{C}_{\mathcal{F}}(X,Y) & \xrightarrow{\mathcal{F}(F)\otimes\mathcal{F}(F)} & \mathcal{D}_{\mathcal{F}}(Y',Z') \otimes \mathcal{D}_{\mathcal{F}}(X',Y') & \xrightarrow{\mathcal{F}(G)\otimes\mathcal{F}(G)} & \mathcal{E}_{\mathcal{F}}(Y'',Z'') \otimes \mathcal{E}_{\mathcal{F}}(X'',Y'') \\
{\scriptstyle \epsilon}\downarrow & & {\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle \epsilon} \\
\mathcal{F}(\mathcal{C}(Y,Z) \otimes \mathcal{C}(X,Y)) & \xrightarrow{\mathcal{F}(F\otimes F)} & \mathcal{F}(\mathcal{D}(Y',Z') \otimes \mathcal{D}(X',Y')) & \xrightarrow{\mathcal{F}(G\otimes G)} & \mathcal{F}(\mathcal{E}(Y'',Z'') \otimes \mathcal{E}(X'',Y'')) \\
{\scriptstyle \mathcal{F}(\bullet)}\downarrow & \Swarrow{\scriptstyle \mathcal{F}(F_\bullet)} & {\scriptstyle \mathcal{F}(\bullet)}\downarrow & \Swarrow{\scriptstyle \mathcal{F}(G_\bullet)} & \downarrow{\scriptstyle \mathcal{F}(\bullet)} \\
\mathcal{C}_{\mathcal{F}}(X,Z) & \xrightarrow[\mathcal{F}(F)]{} & \mathcal{D}_{\mathcal{F}}(X',Z') & \xrightarrow[\mathcal{F}(G)]{} & \mathcal{E}_{\mathcal{F}}(X'',Z'').
\end{array}$$

Similarly, the fact that the unity constraint is compatible with composition amounts to the following observation: First applying $\mathcal{F}$ to the 2-cell given by diagram (2.1) and then precomposing $u\colon \mathbb{1} \to \mathcal{F}(\mathbb{1})$ is the same as first applying $\mathcal{F}$ and precomposing u for both triangles individually before horizontally composing them. □

Example 2.5.7. In Example 2.4.2 we have considered $\mathcal{B}$-enriched categories as $\mathcal{Cat}$-enriched categories, where $\mathcal{B} \subset (\mathcal{Cat}, \times, \{*\})$ is a monoidal 2-subcategory. This is just the functoriality of changing the enrichment, applied to the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{Cat}$.
More generally, for any monoidal 2-category $(\mathcal{V}, \otimes, \mathbb{1})$ and monoidal 2-subcategory $\mathcal{B} \subset \mathcal{V}$, the inclusion functor $\iota\colon \mathcal{B} \hookrightarrow \mathcal{V}$ constitutes a changer (ι, ϵ, u) with ϵ and u being identities. Changing the enrichment then allows us to consider $\mathcal{B}\text{-}\mathcal{Cat}$ as a subcategory of $\mathcal{V}\text{-}\mathcal{Cat}$.
A lax $\mathcal{V}$-enriched functor $F\colon C \to \mathcal{D}$ (with C and $\mathcal{D}$ lying in $\mathcal{B}\text{-}\mathcal{Cat}$) is a lax $\mathcal{B}$-enriched functor if and only if all its data lies in $\mathcal{B}$:

$$F(X,Y)\colon C(X,Y) \to \mathcal{D}(F(X), F(Y)) \in \mathcal{B} \quad \forall X, Y \in \mathrm{Ob}(C), \qquad F_{\mathrm{id}} \in \mathcal{B}, \qquad F_\bullet \in \mathcal{B}.$$

The process of changing the enrichment $\mathrm{ChEn}(C, \mathcal{F})$ depends on two kinds of data, namely a changer $(\mathcal{F}\colon \mathcal{V} \to \mathcal{W}, \epsilon, u)$ and a $\mathcal{V}$-enriched category $C \in \mathcal{V}\text{-}\mathcal{Cat}$. In Theorem 2.5.6, we have shown that ChEn is functorial in C. Our next goal is to study in which sense ChEn is functorial in the changer. This requires a notion of morphism between changers.

Definition 2.5.8. Let $(\mathcal{F}\colon \mathcal{V} \to \mathcal{W}, \epsilon, u)$ and $(\mathcal{G}\colon \mathcal{V} \to \mathcal{W}, \delta, v)$ be two changers between monoidal 2-categories $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$. A **monoidal 2-natural transformation** $\beta\colon \mathcal{F} \Rightarrow \mathcal{G}$ is a 2-natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$, which respects the monoidal structure in the sense that for all objects $X, Y \in \mathrm{Ob}(\mathcal{V})$, the following two diagrams in $\mathcal{W}$ commute:

$$\begin{array}{ccc} \mathcal{F}(X) \otimes \mathcal{F}(Y) & \xrightarrow{\beta \otimes \beta} & \mathcal{G}(X) \otimes \mathcal{G}(Y) \\ {\scriptstyle \epsilon}\downarrow & & \downarrow{\scriptstyle \delta} \\ \mathcal{F}(X \otimes Y) & \xrightarrow{\ \ \beta\ \ } & \mathcal{G}(X \otimes Y) \end{array} \qquad \begin{array}{ccc} & \mathbb{1} & \\ {\scriptstyle u}\swarrow & & \searrow{\scriptstyle v} \\ \mathcal{F}(\mathbb{1}) & \xrightarrow{\ \ \beta\ \ } & \mathcal{G}(\mathbb{1}). \end{array}$$

Definition 2.5.9. Let $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$ be monoidal 2-categories. The changers (lax monoidal functors of 2-categories) $\mathcal{V} \to \mathcal{W}$ form a category $\mathit{2Cat}_{\otimes}^{\mathit{lax}}(\mathcal{V}, \mathcal{W})$ with morphisms given by the monoidal 2-natural transformations. Composition is defined by composing the underlying 2-natural transformations.

Theorem 2.5.10 (Functoriality of Change of Enrichment II). Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a monoidal 2-category and C a $\mathcal{V}$-enriched category. Construction 2.4.1 constitutes a functor

$$\mathrm{ChEn}(C, -)\colon \mathit{2Cat}_{\otimes}^{\mathit{lax}}(\mathcal{V}, \mathcal{W}) \to \mathcal{W}\text{-}\mathcal{Cat}, \ \mathcal{F} \mapsto C_{\mathcal{F}}, \ \beta \mapsto C_\beta.$$

The strict $\mathcal{W}$-enriched functor $C_\beta\colon C_{\mathcal{F}} \to C_{\mathcal{G}}$ induced by $\beta\colon \mathcal{F} \Rightarrow \mathcal{G}$ is defined as follows:

1. It is the identity $\mathrm{Ob}(C_{\mathcal{F}}) = \mathrm{Ob}(C) \to \mathrm{Ob}(C) = \mathrm{Ob}(C_{\mathcal{G}})$ on objects.

2. For $X, Y \in \mathrm{Ob}(C)$, its action on 1-cells is

$$\beta\colon \mathcal{F}(C(X,Y)) \to \mathcal{G}(C(X,Y)).$$

Proof. For $X, Y, Z \in \mathrm{Ob}(C)$, the commuting diagrams

$$\begin{array}{ccccc} \mathbb{1} & \xrightarrow{\ v\ } & \mathcal{G}(\mathbb{1}) & \xrightarrow{\mathcal{G}(\mathrm{id}^C)} & \mathcal{G}(C(X,X)) \\ & {\scriptstyle u}\searrow & \uparrow{\scriptstyle \beta} & & {\scriptstyle \beta}\uparrow \\ & & \mathcal{F}(\mathbb{1}) & \xrightarrow{\mathcal{F}(\mathrm{id}^C)} & \mathcal{F}(C(X,X)) \end{array}$$

$$\begin{array}{ccc}
\mathcal{F}(C(Y,Z)) \otimes \mathcal{F}(C(X,Y)) & \xrightarrow{\beta\otimes\beta} & \mathcal{G}(C(Y,Z)) \otimes \mathcal{G}(C(X,Y)) \\
{\scriptstyle\epsilon}\big\downarrow & & \big\downarrow{\scriptstyle\delta} \\
\mathcal{F}(C(Y,Z) \otimes C(X,Y)) & \xrightarrow{\quad\beta\quad} & \mathcal{G}(C(Y,Z) \otimes C(X,Y)) \\
{\scriptstyle\mathcal{F}(\bullet)}\big\downarrow & & \big\downarrow{\scriptstyle\mathcal{G}(\bullet)} \\
\mathcal{F}(C(X,Z)) & \xrightarrow{\qquad\beta\qquad} & \mathcal{G}(C(X,Z))
\end{array}$$

show that C_β is a strict functor. It is immediate that $\mathrm{ChEn}(C,-)$ respects identity and composition, so it is a functor. □

Combining our two functoriality results Theorem 2.5.6 and Theorem 2.5.10, we obtain the functor

$$\mathcal{V}\text{-}\mathcal{C}at \times 2\mathcal{C}at_{\otimes}^{lax}(\mathcal{V},\mathcal{W}) \to \mathcal{W}\text{-}\mathcal{C}at$$

and thus the following result.

Theorem 2.5.11 (Functoriality of Change of Enrichment III). For two fixed monoidal 2-categories $(\mathcal{V},\otimes,\mathbb{1})$ and $(\mathcal{W},\otimes,\mathbb{1})$, change of enrichment constitutes a functor

$$\mathrm{ChEn}\colon 2\mathcal{C}at_{\otimes}^{lax}(\mathcal{V},\mathcal{W}) \to [\mathcal{V}\text{-}\mathcal{C}at, \mathcal{W}\text{-}\mathcal{C}at],\ \mathcal{F} \mapsto \mathrm{ChEn}(-,\mathcal{F}),\ \beta \mapsto \mathrm{ChEn}(-,\beta),$$

where $\mathrm{ChEn}(-,\mathcal{F})$ is defined as in Theorem 2.5.6. For a monoidal 2-natural transformation $\beta\colon \mathcal{F} \Rightarrow \mathcal{G}$, the natural transformation $\mathrm{ChEn}(-,\beta)\colon \mathrm{ChEn}(-,\mathcal{F}) \Rightarrow \mathrm{ChEn}(-,\mathcal{G})$ has components $C_\beta\colon C_{\mathcal{F}} \to C_{\mathcal{G}}$ as specified in Theorem 2.5.10.

2.6 2-Functoriality of Change of Enrichment

In the previous section, we have understood change of enrichment as a functor $\mathrm{ChEn}\colon 2\mathcal{C}at_{\otimes}^{lax}(\mathcal{V},\mathcal{W}) \to [\mathcal{V}\text{-}\mathcal{C}at, \mathcal{W}\text{-}\mathcal{C}at]$ for fixed monoidal 2-categories $(\mathcal{V},\otimes,\mathbb{1})$ and $(\mathcal{W},\otimes,\mathbb{1})$. In this section, we improve this result by establishing change of enrichment as a 2-functor

$$2\mathcal{C}at_{\otimes}^{lax} \to \mathcal{C}at$$

from the 2-category of (small) monoidal 2-categories to the 2-category of (small) categories.

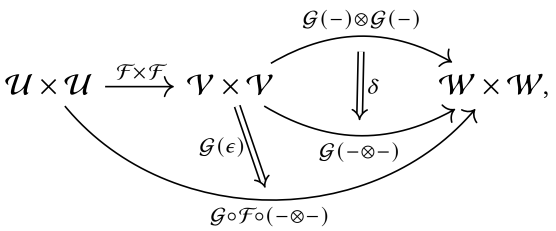
As the first step to making this precise, we need to define the former 2-category. We have already described $2\mathcal{C}at_{\otimes}^{lax}(\mathcal{V},\mathcal{W})$ in Definition 2.5.9, which serve as the hom-categories of the category of monoidal 2-categories.

Definition 2.6.1. The 2-category $2\mathcal{C}at_{\otimes}^{lax}$ consists of the following data:

1. Its objects are the (small) monoidal 2-categories $(\mathcal{V},\otimes,\mathbb{1})$.
2. The hom-categories $2\mathcal{C}at_{\otimes}^{lax}(\mathcal{V},\mathcal{W})$ are given by Definition 2.5.9.
3. The identity 1-cell of $\mathcal{V} \in \mathrm{Ob}(2\mathcal{C}at_{\otimes}^{lax})$ is $(\mathrm{id}_{\mathcal{V}}, \mathrm{id}_{\otimes}, \mathrm{id}_{\mathbb{1}})$.
4. The composition of two changers

$$\mathcal{U} \xrightarrow{(\mathcal{F},\epsilon,u)} \mathcal{V} \xrightarrow{(\mathcal{G},\delta,v)} \mathcal{W}$$

is the changer consisting of the composition of 2-functors $\mathcal{G}\circ\mathcal{F}\colon \mathcal{U} \to \mathcal{W}$ together with



$$\mathbb{1} \xrightarrow{v} \mathcal{G}(\mathbb{1}) \xrightarrow{\mathcal{G}(u)} \mathcal{G}(\mathcal{F}(\mathbb{1})).$$

5. The horizontal composition of monoidal 2-natural transformations is just the horizontal composition of the underlying 2-natural transformations.

$2\mathcal{C}at_{\otimes}^{lax}$ forms a 2-category essentially because the monoidal categories and the 2-categories ($\mathcal{C}at$-enriched categories) each carry the structure of a 2-category.

This allows us to show that change of enrichment is the action on hom-categories of a functor $2\mathcal{C}at_{\otimes}^{lax} \to \mathcal{C}at$, which subsumes our previous functoriality results.

Theorem 2.6.2 (2-**Functoriality of Change of Enrichment**)**.** There exists a strict functor

$$\mathrm{En}\colon 2\mathcal{C}at_{\otimes}^{lax} \to \mathcal{C}at,\ \mathcal{V} \mapsto \mathcal{V}\text{-}\mathcal{C}at,$$

whose action on hom-categories is given by change of enrichment

$$\mathrm{ChEn}\colon 2\mathcal{C}at_{\otimes}^{lax}(\mathcal{V},\mathcal{W}) \to [\mathcal{V}\text{-}\mathcal{C}at, \mathcal{W}\text{-}\mathcal{C}at],\ \mathcal{F} \mapsto \mathrm{ChEn}(-,\mathcal{F}),\ \beta \mapsto \mathrm{ChEn}(-,\beta)$$

as in Theorem 2.5.11.

Proof. We have to show that $\mathrm{En}\colon 2\mathcal{C}at_{\otimes}^{lax} \to \mathcal{C}at$ preserves identities and composition. Given the identity changer $\mathcal{F} = (\mathrm{id}_{\mathcal{V}}, \mathrm{id}_{\otimes}, \mathrm{id}_{\mathbb{1}})$ of a monoidal 2-category $\mathcal{V}$, its image under change of enrichment is the functor

$$\mathcal{V}\text{-}\mathcal{C}at \to \mathcal{V}\text{-}\mathcal{C}at,\ C \mapsto C_{\mathcal{F}},\ G \mapsto G_{\mathcal{F}}.$$

By unpacking the definition of $C_{\mathcal{F}}$ and $G_{\mathcal{F}}$ in Construction 2.4.1 and Theorem 2.5.6, respectively, it becomes apparent that this is the identity functor.

More work is required to see that composition is also preserved. To that end, consider changers

$$\mathcal{U} \xrightarrow{(\mathcal{F},\epsilon,u)} \mathcal{V} \xrightarrow{(\mathcal{G},\delta,v)} \mathcal{W}.$$

We claim that the functors $\mathcal{U}\text{-}\mathcal{C}at \to \mathcal{W}\text{-}\mathcal{C}at$

$$\mathrm{ChEn}(-,\mathcal{G}) \circ \mathrm{ChEn}(-,\mathcal{F}) = \mathrm{ChEn}(-,\mathcal{G}\circ\mathcal{F}) \tag{2.2}$$

are equal. We first check that the two functors agree on objects. Fixing an object $C \in \mathcal{U}\text{-}\mathcal{C}at$, the left functor sends this $\mathcal{U}$-enriched category to the $\mathcal{W}$-enriched category $(C_{\mathcal{F}})_{\mathcal{G}}$, whereas the right functor sends it to $C_{\mathcal{G}\circ\mathcal{F}}$. The objects of both of these $\mathcal{W}$-enriched categories are those of C. Similarly, for objects $X, Y \in \mathrm{Ob}(C)$, we have

$$(C_{\mathcal{F}})_{\mathcal{G}}(X,Y) = \mathcal{G}(\mathcal{F}(C(X,Y))) = C_{\mathcal{G}\circ\mathcal{F}}$$

by Construction 2.4.1 and Definition 2.6.1. The identity 1-cell is

$$\mathbb{1} \xrightarrow{v} \mathcal{G}(\mathbb{1}) \xrightarrow{\mathcal{G}(u)} \mathcal{G}(\mathcal{F}(\mathbb{1})) \xrightarrow{\mathcal{G}(\mathcal{F}(\mathrm{id}^{C}))} \mathcal{G}(\mathcal{F}(C(X,X)))$$

and the equality of horizontal composition, unitors and associator is also straightforward to see, showing that the two functors (2.2) are equal on objects.

For a lax $\mathcal{U}$-enriched functor $H\colon C \to \mathcal{D}$, both functors (2.2) yield a lax $\mathcal{W}$-enriched functor and from their description in Theorem 2.5.6, we see that they are equal. Indeed, on objects both functors are given by H and their local action is

$$\mathcal{G}(\mathcal{F}(H))\colon \mathcal{G}(\mathcal{F}(C(X,Y))) \to \mathcal{G}\big(\mathcal{F}\big(\mathcal{D}\big(H(X),H(Y)\big)\big)\big).$$

Their unity constraint is the 2-cell

$$\begin{array}{ccccccc} \mathbb{1} \xrightarrow{v} \mathcal{G}(\mathbb{1}) \xrightarrow{\mathcal{G}(u)} & \mathcal{G}(\mathcal{F}(\mathbb{1})) & & \xrightarrow{\quad \mathcal{G}(\mathcal{F}(\mathrm{id}^{\mathcal{D}})) \quad} & & \mathcal{G}(\mathcal{F}(\mathcal{D}(H(X),H(X)))) \\ & & \searrow^{\mathcal{G}(\mathcal{F}(\mathrm{id}^{C}))} & \Big\Downarrow{\scriptstyle \mathcal{G}(\mathcal{F}(H_{\mathrm{id}}))} & \nearrow_{\mathcal{G}(\mathcal{F}(H))} & \\ & & & \mathcal{G}(\mathcal{F}(C(X,X))) & & \end{array}$$

and their functoriality constraint is

$$\begin{array}{ccc}
\mathcal{C}_{\mathcal{G}\circ\mathcal{F}}(Y,Z)\otimes\mathcal{C}_{\mathcal{G}\circ\mathcal{F}}(X,Y) & \xrightarrow{\mathcal{G}(\mathcal{F}(H))\otimes\mathcal{G}(\mathcal{F}(H))} & \mathcal{D}_{\mathcal{G}\circ\mathcal{F}}(H(Y),H(Z))\otimes\mathcal{D}_{\mathcal{G}\circ\mathcal{F}}(H(X),H(Y)) \\
\delta\downarrow & & \downarrow\delta \\
\mathcal{G}(\mathcal{C}_{\mathcal{F}}(Y,Z)\otimes\mathcal{C}_{\mathcal{F}}(X,Y)) & \xrightarrow{\mathcal{G}(\mathcal{F}(H)\otimes\mathcal{F}(H))} & \mathcal{G}(\mathcal{D}_{\mathcal{F}}(H(Y),H(Z))\otimes\mathcal{D}_{\mathcal{F}}(H(X),H(Y))) \\
\mathcal{G}(\epsilon)\downarrow & & \downarrow\mathcal{G}(\epsilon) \\
\mathcal{G}(\mathcal{F}(\mathcal{C}(Y,Z)\otimes\mathcal{C}(X,Y))) & \xrightarrow{\mathcal{G}(\mathcal{F}(H\otimes H))} & \mathcal{G}(\mathcal{F}(\mathcal{D}(H(Y),H(Z))\otimes\mathcal{D}(H(X),H(Y)))) \\
\mathcal{G}(\mathcal{F}(\bullet))\downarrow & \Swarrow\ \mathcal{G}(\mathcal{F}(H_\bullet)) & \downarrow\mathcal{G}(\mathcal{F}(\bullet)) \\
\mathcal{C}_{\mathcal{G}\circ\mathcal{F}}(X,Z) & \xrightarrow[\mathcal{G}(\mathcal{F}(H))]{} & \mathcal{D}_{\mathcal{G}\circ\mathcal{F}}(H(X),H(Z)).
\end{array}$$

This shows that composition of changers is preserved and hence it remains to verify that the composition of monoidal 2-natural transformations

$$\mathcal{U}\ \underset{(\mathcal{F}',\epsilon',u')}{\overset{(\mathcal{F},\epsilon,u)}{\Downarrow\alpha}}\ \mathcal{V}\ \underset{(\mathcal{G}',\delta',v')}{\overset{(\mathcal{G},\delta,v)}{\Downarrow\beta}}\ \mathcal{W}$$

is preserved. First horizontally composing these two monoidal 2-natural transformations yields the monoidal 2-natural transformation $\mathcal{G}\circ\mathcal{F}\Rightarrow\mathcal{G}'\circ\mathcal{F}'$ with components (for an object $C\in\mathcal{U}$)

$$\mathcal{G}(\mathcal{F}(C))\xrightarrow{\mathcal{G}(\alpha)}\mathcal{G}(\mathcal{F}'(C))\xrightarrow{\beta}\mathcal{G}'(\mathcal{F}'(C)).$$

By Theorem 2.5.10, its image under $\mathrm{En}\colon \mathit{2Cat}_{\otimes}^{lax}\to\mathit{Cat}$

$$\mathcal{U}\text{-}\mathit{Cat}\ \underset{\mathrm{ChEn}(-,\mathcal{G}'\circ\mathcal{F}')}{\overset{\mathrm{ChEn}(-,\mathcal{G}\circ\mathcal{F})}{\Downarrow\,\mathrm{ChEn}(-,\beta\circ\alpha)}}\ \mathcal{W}\text{-}\mathit{Cat}$$

has as components the strict identity-on-objects $\mathcal{W}$-enriched functor

$$\mathcal{C}_{\mathcal{G}\circ\mathcal{F}}\to\mathcal{C}_{\mathcal{G}'\circ\mathcal{F}'},\qquad \mathcal{G}(\mathcal{F}(\mathcal{C}(X,Y)))\xrightarrow{\mathcal{G}(\alpha)}\mathcal{G}(\mathcal{F}'(\mathcal{C}(X,Y)))\xrightarrow{\beta}\mathcal{G}'(\mathcal{F}'(\mathcal{C}(X,Y))).$$

On the other hand, the horizontal composition

$$\mathcal{U}\text{-}\mathit{Cat}\ \underset{\mathrm{ChEn}(-,\mathcal{F}')}{\overset{\mathrm{ChEn}(-,\mathcal{F})}{\Downarrow\,\mathrm{ChEn}(-,\alpha)}}\ \mathcal{V}\text{-}\mathit{Cat}\ \underset{\mathrm{ChEn}(-,\mathcal{G}')}{\overset{\mathrm{ChEn}(-,\mathcal{G})}{\Downarrow\,\mathrm{ChEn}(-,\beta)}}\ \mathcal{W}\text{-}\mathit{Cat}$$

in Cat has components

$$(\mathcal{C}_{\mathcal{F}})_{\mathcal{G}}\xrightarrow{\mathrm{ChEn}(-,\mathcal{G})(\mathcal{C}_\alpha)}(\mathcal{C}_{\mathcal{F}'})_{\mathcal{G}}\xrightarrow{(\mathcal{C}_{\mathcal{F}'})_\beta}(\mathcal{C}_{\mathcal{F}'})_{\mathcal{G}'}.$$

By definition of $\mathrm{ChEn}(-,\mathcal{G})$ (see Theorem 2.5.6), the identity-on-objects strict $\mathcal{V}$-enriched functor $\mathcal{C}_\alpha$ yields an identity-on-objects strict $\mathcal{W}$-enriched functor. By unraveling the definition of $\mathrm{ChEn}(-,\mathcal{G})$, $\mathcal{C}_\alpha$ and $(\mathcal{C}_{\mathcal{F}'})_\beta$, we conclude that this composition is the same $\mathcal{W}$-enriched functor as above, proving the theorem. □

As was the case for the 1-categorical version Theorem 2.1.4, it follows that adjunctions are preserved by En: $\mathit{2Cat}_{\otimes}^{\mathit{lax}} \to \mathit{Cat}$.

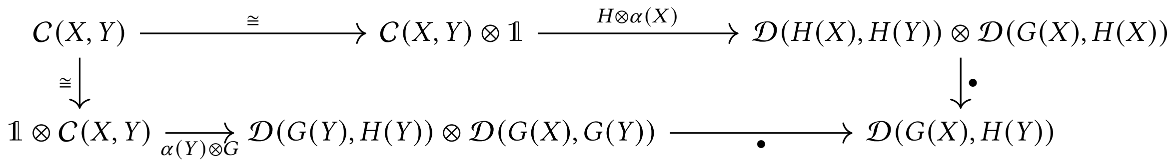
Remark 2.6.3. By Theorem 2.1.4, in the setting of ordinary monoidal categories, change of enrichment yields a 2-functor

$$\mathit{Cat}_{\otimes}^{\mathit{lax}} \to \mathit{2Cat},\ \mathcal{V} \mapsto \mathcal{V}\text{-}\mathit{Cat},$$

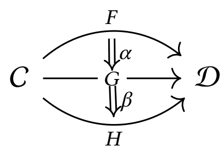
where the 2-category of (small) $\mathcal{V}$-enriched categories $\mathcal{V}\text{-}\mathcal{C}\mathit{at}$ has the $\mathcal{V}$-enriched natural transformations as 2-cells. Recall that a *$\mathcal{V}$-enriched natural transformation* $\alpha\colon G \Rightarrow H$ consists of 1-cells in $\mathcal{V}$

$$\alpha(X)\colon \mathbb{1} \to \mathcal{D}(G(X), H(X))$$

for every object $X \in \mathcal{V}$ (called *components* of α) which are natural in the sense that the diagram



in $\mathcal{V}$ commutes for all $X, Y \in \mathcal{V}$.
The vertical composition of $\mathcal{V}$-enriched natural transformations



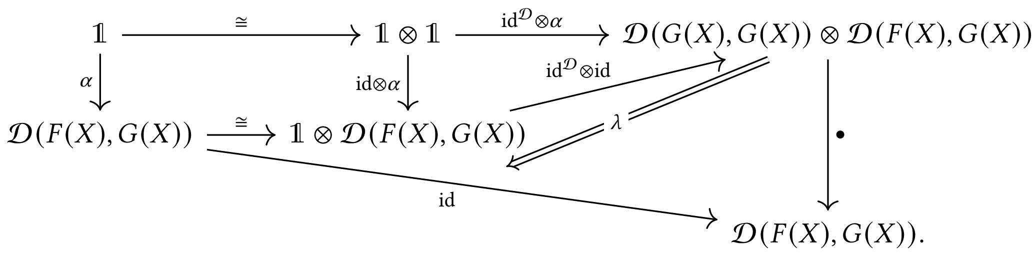
has components

$$\mathbb{1} \xrightarrow{\cong} \mathbb{1} \otimes \mathbb{1} \xrightarrow{\beta\otimes\alpha} \mathcal{D}(G(X), H(X)) \otimes \mathcal{D}(F(X), G(X)) \xrightarrow{\bullet} \mathcal{D}(F(X), H(X)).$$

Using this definition verbatim for categories enriched over a monoidal 2-category $(\mathcal{V}, \otimes, \mathbb{1})$, it becomes apparent that $\mathcal{V}\text{-}\mathcal{C}\mathit{at}$ is not a 2-category (or bicategory) anymore, because its "hom-categories" $\mathcal{V}\text{-}\mathcal{C}\mathit{at}(C, \mathcal{D})$ are not ordinary categories. For instance, the identity $\mathcal{V}$-enriched natural transformation id: $G \Rightarrow G$

$$\mathrm{id}(X)\colon \mathbb{1} \xrightarrow{\mathrm{id}^{\mathcal{D}}} \mathcal{D}(G(X), G(X))$$

is not a strict identity w.r.t. vertical composition, but only an identity "up to invertible 2-cell" (where the regions in the diagram without a 2-cell commute):



This demonstrates that higher-categorical tools are required to properly generalize Theorem 2.1.4.
We only sketch the basic idea. Fixing a changer $(\mathcal{F}\colon \mathcal{V} \to \mathcal{W}, \epsilon, u)$ between monoidal 2-categories $(\mathcal{V}, \otimes, \mathbb{1})$ and $(\mathcal{W}, \otimes, \mathbb{1})$, the first step is to extend the functor

$$\mathrm{ChEn}(-, \mathcal{F})\colon \mathcal{V}\text{-}\mathcal{C}\mathit{at} \to \mathcal{W}\text{-}\mathcal{C}\mathit{at},\ C \mapsto C_{\mathcal{F}},\ G \mapsto G_{\mathcal{F}}$$

from Theorem 2.5.6 by defining its action on 2-cells to be

$$(\alpha\colon G \Rightarrow H) \quad\longmapsto\quad \big(\mathrm{ChEn}(\alpha,\mathcal{F})(X)\colon \mathbb{1} \xrightarrow{u} \mathcal{F}(\mathbb{1}) \xrightarrow{\mathcal{F}(\alpha(X))} \mathcal{D}_{\mathcal{F}}(G(X),H(X))\big).$$

This definition preserves the horizontal composition of 2-cells. Indeed, the identity $\mathcal{V}$-enriched natural transformation $\mathrm{id}\colon G \Rightarrow G$ gets mapped to the $\mathcal{W}$-enriched natural transformation with components

$$\mathbb{1} \xrightarrow{u} \mathcal{F}(\mathbb{1}) \xrightarrow{\mathcal{F}(\mathrm{id}^{\mathcal{D}})} \mathcal{D}_{\mathcal{F}}(G(X),G(X)),$$

which by definition of $\mathcal{D}_{\mathcal{F}}$ is the identity $\mathcal{W}$-enriched natural transformation $\mathrm{id}\colon G_{\mathcal{F}} \Rightarrow G_{\mathcal{F}}$.
Similarly, for two $\mathcal{V}$-enriched natural transformations

$$\mathcal{C} \underset{H}{\overset{G}{\Downarrow \alpha}} \mathcal{D} \underset{H'}{\overset{G'}{\Downarrow \beta}} \mathcal{E},$$

the components of $\mathrm{ChEn}(\beta\circ\alpha,\mathcal{F})$ are given by the composition

$$\mathbb{1} \xrightarrow{\mathcal{F}(G')\circ\mathcal{F}(\alpha)\circ u} \mathcal{E}_{\mathcal{F}}(G'G(X),G'H(X)) \xrightarrow{\mathcal{F}(\beta\otimes\mathrm{id})} \mathcal{F}(\dots) \xrightarrow{\mathcal{F}(\bullet)} \mathcal{E}_{\mathcal{F}}(G'G(X),H'H(X))$$

and those of $\mathrm{ChEn}(\beta,\mathcal{F})\circ\mathrm{ChEn}(\alpha,\mathcal{F})$ are

$$\mathbb{1} \xrightarrow{\mathcal{F}(G')\circ\mathcal{F}(\alpha)\circ u} \mathcal{E}_{\mathcal{F}}(G'G(X),G'H(X)) \xrightarrow{(\mathcal{F}(\beta)\circ u)\otimes\mathrm{id}} \mathcal{E}_{\mathcal{F}}(\dots)\otimes\mathcal{E}_{\mathcal{F}}(\dots) \xrightarrow{\hat{\bullet}} \mathcal{E}_{\mathcal{F}}(G'G(X),H'H(X)),$$

where we have for brevity omitted some of the objects. Expanding the definition of $\hat{\bullet}$ and canceling the common beginning and end of the compositions, it suffices to prove that

$$\mathcal{E}_{\mathcal{F}}(G'G(X),G'H(X)) \xrightarrow{\mathcal{F}(\cong)} \mathcal{F}(\mathbb{1}\otimes\mathcal{E}(G'G(X),G'H(X))) \xrightarrow{\mathcal{F}(\beta\otimes\mathrm{id})} \mathcal{F}(\dots\otimes\dots)$$

is equal to

$$\mathcal{E}_{\mathcal{F}}(G'G(X),G'H(X)) \cong \mathbb{1}\otimes\mathcal{E}_{\mathcal{F}}(\dots) \xrightarrow{u\otimes\mathrm{id}} \mathcal{F}(\mathbb{1})\otimes\mathcal{E}_{\mathcal{F}}(\dots) \xrightarrow{\mathcal{F}(\beta)\otimes\mathrm{id}} \mathcal{E}_{\mathcal{F}}(\dots)\otimes\mathcal{E}_{\mathcal{F}}(\dots) \xrightarrow{\epsilon} \mathcal{F}(\dots)$$

and this follows from the naturality of ϵ and the unitality of $\mathcal{F}$. ○

3 Change of Enrichment for represented Changers

For a given category C, represented functors $\mathrm{Hom}(C,-)\colon C \to \mathcal{S}et$ play a fundamental role for ordinary categories. Since any monoidal 2-categories $(\mathcal{V},\otimes,\mathbb{1})$ is by definition $\mathcal{C}at$-enriched, we obtain represented 2-functors

$$\mathcal{V}(C,-)\colon \mathcal{V} \to \mathcal{C}at$$

and as in the 1-dimensional case, these are particularly easy to understand. In this chapter, we characterize when a represented 2-functor $\mathcal{V}(C,-)\colon \mathcal{V} \to \mathcal{C}at$ is a changer. It turns out that this is equivalent to the structure of a comonoid on the representing object C (see Theorem 3.1.3). This result allows us to give some key examples, like the underlying bicategory functor (see Definition 3.2.2).

3.1 Represented Changers

In an arbitrary small 2-category (i.e. $\mathcal{C}at$-enriched category) $\mathcal{V}$, the functors $\mathcal{V}(C,-)\colon \mathcal{V} \to \mathcal{C}at$

$$X \underset{h}{\overset{g}{\Downarrow\alpha}} Y \quad\longmapsto\quad \mathcal{V}(C,X) \underset{h\circ-}{\overset{g\circ-}{\Downarrow\alpha\circ-}} \mathcal{V}(C,Y),$$

represented by a fixed object $C \in \mathcal{V}$ offer an important source of functors $\mathcal{V} \to \mathcal{C}at$. If $(\mathcal{V},\otimes,\mathbb{1})$ is a monoidal 2-category, it is thus natural to consider changing the enrichment w.r.t. such a functor, which would yield a functor $\mathcal{V}\text{-}\mathcal{C}at \to \mathcal{B}icat$.

However, this requires $\mathcal{V}(C,-)\colon (\mathcal{V},\otimes,\mathbb{1}) \to (\mathcal{C}at,\times,\{*\})$ to be a changer (i.e. to be lax monoidal), so additional structure is required. This extra structure is precisely that of a comonoid on the representing object $C \in \mathcal{V}$, as we will see. Before we state and prove the corresponding theorem, we recall the enriched Yoneda lemma (see e.g. [Kel05, Sec 1.9]) for $\mathcal{C}at$.

Lemma 3.1.1 (**$\mathcal{C}at$-enriched Yoneda lemma**)**.** Let $\mathcal{V}$ be a small 2-category and $F\colon \mathcal{V} \to \mathcal{C}at$ be a 2-functor. A 2-natural transformation $\alpha\colon \mathcal{V}(C,-) \Rightarrow F$ is uniquely determined by its value at the identity id_C; that is, by the object $\alpha(C)(\mathrm{id}_C) \in F(C)$.
On the other hand, any object $c \in F(C)$ induces a 2-natural transformation $\alpha\colon \mathcal{V}(C,-) \Rightarrow F$ with components

$$\alpha(X)\colon\ \mathcal{V}(C,X) \xrightarrow{F} [F(C),F(X)] \xrightarrow{\mathrm{eval}_c} F(X).$$

This yields a bijection between objects $c \in F(C)$ and 2-natural transformations $\alpha\colon \mathcal{V}(C,-) \Rightarrow F$.

This is sometimes called the *weak enriched Yoneda lemma*, referring to the fact that there is also a stronger version, stating that the above bijection in fact constitutes an isomorphism of categories.

The following special case is obtained from Lemma 3.1.1 by observing that the functor represented by $(C,D) \in \mathcal{V}\times\mathcal{V}$ is $\mathcal{V}(C,-)\times\mathcal{V}(D,-)\colon \mathcal{V}\times\mathcal{V} \to \mathcal{C}at$.

Lemma 3.1.2. Let $\mathcal{V}$ be a small 2-category and $F\colon \mathcal{V}\times\mathcal{V} \to \mathcal{C}at$ be a 2-functor. There is a bijection between the set of 2-natural transformations $\alpha\colon \mathcal{V}(C,-)\times\mathcal{V}(D,-) \Rightarrow F$ and objects $c \in F(C,D)$. It maps a 2-natural transformation $\alpha\colon \mathcal{V}(C,-)\times\mathcal{V}(D,-) \Rightarrow F$ to $\alpha(C,D)(\mathrm{id}_C,\mathrm{id}_D) \in F(C,D)$ and an object $c \in F(C,D)$ to the 2-natural transformation with components

$$\alpha(X,Y)\colon\ \mathcal{V}(C,X)\times\mathcal{V}(D,Y) \xrightarrow{F} [F(C,D),F(X,Y)] \xrightarrow{\mathrm{eval}_c} F(X,Y).$$

Theorem 3.1.3 (Correspondence between represented Changers and Comonoids). Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a small monoidal 2-category, $C \in \mathcal{V}$ be an object and consider the functor

$$\mathcal{V}(C, -)\colon (\mathcal{V}, \otimes, \mathbb{1}) \to (\mathcal{C}at, \times, \{*\})$$

represented by C. The structure of a comonoid $(C,\ \mu\colon C \to C \otimes C,\ \eta\colon C \to \mathbb{1})$ on $C \in \mathcal{V}$ makes $\mathcal{V}(C, -)$ into a changer $(\mathcal{V}(C, -), \delta, v)$, where the 2-natural transformation $\delta\colon \mathcal{V}(C, -) \times \mathcal{V}(C, -) \Rightarrow \mathcal{V}(C, - \otimes -)$ has components

$$\delta(X, Y)\colon\ \mathcal{V}(C, X) \times \mathcal{V}(C, Y) \xrightarrow{\ \otimes\ } \mathcal{V}(C \otimes C, X \otimes Y) \xrightarrow{-\circ\mu} \mathcal{V}(C, X \otimes Y).$$

and $v\colon \{*\} \to \mathcal{V}(C, \mathbb{1})$ is the functor choosing η.
Conversely, a changer $(\mathcal{V}(C, -), \delta, v)$ yields a comonoid (C, μ, η) on $C \in \mathcal{C}$ via

$$\delta(C, C)(\mathrm{id}_C, \mathrm{id}_C)\colon C \to C \otimes C, \qquad \eta = v(*).$$

This yields a bijection between changers represented by C and comonoid structures on C.

Proof. First note that the component $\delta(X, Y)$ can equivalently be written as the composition

$$\mathcal{V}(C, X) \times \mathcal{V}(C, Y) \xrightarrow{\mathcal{V}(C, -\otimes-)} [\mathcal{V}(C, C \otimes C), \mathcal{V}(C, X \otimes Y)] \xrightarrow{\mathrm{eval}_\mu} \mathcal{V}(C, X \otimes Y),$$

which is exactly the 2-natural transformation corresponding to μ in Lemma 3.1.2. It follows that δ is 2-natural and that the above constitutes a bijection between triples $(\mathcal{V}(C, -), \delta, v)$ and (C, μ, η). Consequently, we have to show that $(\mathcal{V}(C, -), \delta, v)$ is a lax monoidal functor on the underlying monoidal categories if and only if (C, μ, η) is a comonoid in them.
Denoting the left $\mathcal{V}$-associator by λ, one of the unitality diagrams of $(\mathcal{V}(C, -), \delta, v)$ is

$$\begin{array}{ccc}
\{*\} \times \mathcal{V}(C, X) & \xrightarrow{v \times \mathrm{id}} & \mathcal{V}(C, \mathbb{1}) \times \mathcal{V}(C, X) \\
\Big\downarrow{\cong} & & \Big\downarrow{\otimes} \\
 & & \mathcal{V}(C \otimes C, \mathbb{1} \otimes X) \\
 & & \Big\downarrow{-\circ\mu} \\
\mathcal{V}(C, X) & \xleftarrow{\lambda\circ-} & \mathcal{V}(C, \mathbb{1} \otimes X).
\end{array}$$

This diagram commutes if and only if every 1-cell $f\colon C \to X$ is equal to the composition

$$C \xrightarrow{\ \mu\ } C \otimes C \xrightarrow{\eta\otimes f} \mathbb{1} \otimes X \xrightarrow{\ \lambda\ } X,$$

which by naturality of λ is equal to

$$C \xrightarrow{\ \mu\ } C \otimes C \xrightarrow{\eta\otimes\mathrm{id}} \mathbb{1} \otimes C \xrightarrow{\ \lambda\ } C \xrightarrow{\ f\ } X.$$

This equality being true for every $f\colon C \to X$ is equivalent to it being true for the identity $\mathrm{id}\colon C \to C$

$$C \xrightarrow{\ \mu\ } C \otimes C \xrightarrow{\eta\otimes\mathrm{id}} \mathbb{1} \otimes C \xrightarrow{\ \lambda\ } C \quad = \quad C \xrightarrow{\ \mathrm{id}\ } C,$$

which is exactly the left unitality of (C, μ, η). An analogous argument applies to the other unitality diagram.
Writing α for the $\mathcal{V}$-associator, we argue analogously for the associativity of $(\mathcal{V}(C, -), \delta, v)$. Its associativity translates to the statement that for 1-cells $f\colon C \to X$, $g\colon C \to Y$ and $h\colon C \to Z$, the diagram

$$\begin{array}{ccccccccc}
C & \xrightarrow{\ \mu\ } & C \otimes C & \xrightarrow{\mathrm{id}\otimes h} & C \otimes Z & \xrightarrow{\mu\otimes\mathrm{id}} & (C \otimes C) \otimes Z & \xrightarrow{(f\otimes g)\otimes\mathrm{id}} & (X \otimes Y) \otimes Z \\
 & \searrow^{\mu} & & & & & & & \Big\downarrow{\alpha} \\
 & & C \otimes C & \xrightarrow{f\otimes\mathrm{id}} & X \otimes C & \xrightarrow{\mathrm{id}\otimes\mu} & X \otimes (C \otimes C) & \xrightarrow{\mathrm{id}\otimes(g\otimes h)} & X \otimes (Y \otimes Z)
\end{array}$$

commutes. By functoriality of ⊗, this can be rewritten as

$$\begin{array}{ccccccc}
C & \xrightarrow{\mu} & C\otimes C & \xrightarrow{\mu\otimes\mathrm{id}} & (C\otimes C)\otimes C & \xrightarrow{(f\otimes g)\otimes h} & (X\otimes Y)\otimes Z \\
 & & & \searrow^{\mathrm{id}\otimes\mu} & & & \downarrow{\scriptstyle\alpha} \\
 & & & & C\otimes(C\otimes C) & \xrightarrow{f\otimes(g\otimes h)} & X\otimes(Y\otimes Z).
\end{array}$$

The naturality of α shows that the diagram is commutative if and only if

$$(f\otimes(g\otimes h))\circ\alpha\circ(\mu\otimes\mathrm{id})\circ\mu=(f\otimes(g\otimes h))\circ(\mathrm{id}\otimes\mu)\circ\mu$$

and setting $f=g=h=\mathrm{id}_C$ reveals that this is precisely the associativity condition

$$\alpha\circ(\mu\otimes\mathrm{id})\circ\mu=(\mathrm{id}\otimes\mu)\circ\mu$$

of (C,μ,η). □

The result states that finding a represented changer $\mathcal{V}(C,-)\colon(\mathcal{V},\otimes,\mathbb{1})\to(\mathcal{C}at,\times,\{*\})$ is the same as constructing a comonoid in $\mathcal{V}$, which gives rise to some important changes of enrichment, as we will see in the next section.

Remark 3.1.4. The result is best understood in the context of *Day convolution*, originally developed by Day in [Day70a; Day70b]. Let $(\mathcal{V},\otimes,\mathbb{1})$ be a cocomplete closed symmetric monoidal category (in our case $(\mathcal{V},\otimes,\mathbb{1})=(\mathcal{C}at,\times,\{*\})$) and $(C,\otimes,\mathbb{1})$ be a small $\mathcal{V}$-enriched monoidal category. Then the *Day convolution* on the category $[C,\mathcal{V}]$ of $\mathcal{V}$-enriched functors $C\to\mathcal{V}$

$$*\colon[C,\mathcal{V}]\times[C,\mathcal{V}]\to[C,\mathcal{V}]$$

can be defined for $\mathcal{V}$-enriched functors $F,G\colon C\to\mathcal{V}$ as the enriched left Kan extension

$$\begin{array}{ccc}
C\times C & \xrightarrow{F\boxtimes G} & \mathcal{V} \\
 & \searrow^{\otimes}\ \Downarrow\ \nearrow_{F*G} & \\
 & C, &
\end{array}$$

where

$$\boxtimes\colon\ [C,\mathcal{V}]\times[C,\mathcal{V}]\xrightarrow{\times}[C\times C,\mathcal{V}\times\mathcal{V}]\xrightarrow{\otimes\circ-}[C\times C,\mathcal{V}],\qquad(F,G)\mapsto F(-)\otimes G(-)$$

denotes the *external tensor product* [Ric20, Def 9.8.1]. By the universal property of the left Kan extension, this means that for a $\mathcal{V}$-enriched functor $H\colon C\to\mathcal{V}$, $\mathcal{V}$-enriched natural transformations $F*G\Rightarrow H$ correspond to $\mathcal{V}$-enriched natural transformations $F(-)\otimes G(-)\Rightarrow H(-\otimes-)$.
Equipped with Day convolution and the $\mathcal{V}$-enriched functor $C(\mathbb{1},-)$ represented by the unit object $\mathbb{1}\in C$, the category of $\mathcal{V}$-enriched functors $[C,\mathcal{V}]$ becomes a monoidal category. Moreover, a monoid

$$(F\colon C\to\mathcal{V},\ F*F\Rightarrow F,\ C(\mathbb{1},-)\Rightarrow F)$$

in $[C,\mathcal{V}]$ is the same as a lax monoidal functor

$$(F\colon C\to\mathcal{V},\ \epsilon\colon F(-)\otimes F(-)\Rightarrow F(-\otimes-),\ u\colon\mathbb{1}\Rightarrow F(\mathbb{1}))$$

(see [Day70a, Exa 3.3.2] for the case of commutative monoids)[1]. In fact, there is an equivalence of categories between the category of lax monoidal functors and the category of monoids in $[C,\mathcal{V}]$ w.r.t. Day

[1]This result is e.g. applied in the programming language *Haskell* in order to understand *applicative functors* (represented by the type class `Applicative`), see [RJ17].

convolution [Day70b, Prop 9.8.8].
Appealing to the enriched density formula, one observes that the Yoneda embedding

$$\mathcal{C}^{\mathrm{op}} \hookrightarrow [\mathcal{C}, \mathcal{V}],\ C \mapsto \mathcal{C}(C, -)$$

is a strong monoidal functor (when $[\mathcal{C}, \mathcal{V}]$ is equipped with Day convolution).[2]
This observation puts Theorem 3.1.3 in a greater context. Indeed, a comonoid (C, μ, η) in $\mathcal{C}$ (i.e. a monoid in $\mathcal{C}^{\mathrm{op}}$) yields the monoid (since strong monoidal functors preserve monoids)

$$\Big(\mathcal{C}(C,-),\ \ \mathcal{C}(C,-) * \mathcal{C}(C,-) \cong \mathcal{C}(C\otimes C,-) \overset{-\circ\mu}{\Longrightarrow} \mathcal{C}(C,-),\ \ \mathcal{C}(\mathbb{1},-) \overset{-\circ\eta}{\Longrightarrow} \mathcal{C}(C,-)\Big),$$

which corresponds to a lax monoidal $\mathcal{C}at$-enriched functor and this is the changer from the theorem. That conversely every monoid in $[\mathcal{C}, \mathcal{V}]$ stems from a comonoid in $\mathcal{C}$ is a consequence of the fully faithfulness of the Yoneda embedding. ○

We now consider monoidal 2-natural transformations of the form

$$\alpha\colon (\mathcal{V}(C,-), \delta, v) \Rightarrow (\mathcal{F}, \epsilon, u), \qquad \mathcal{V} \underset{\mathcal{F}}{\overset{\mathcal{V}(C,-)}{\Downarrow \alpha}} \mathcal{C}at.$$

Our interest in these is motivated by the 2-functoriality of change of enrichment (Theorem 2.6.2), which translates such a monoidal 2-natural transformation into a natural transformation

$$\mathcal{V}\text{-}\mathcal{C}at \underset{\mathrm{ChEn}(-,\mathcal{F})}{\overset{\mathrm{ChEn}(-,\mathcal{V}(C,-))}{\Downarrow}} \mathcal{B}icat.$$

For example, if the object C is chosen as the unit object $\mathbb{1} \in \mathcal{V}$, then $\mathrm{ChEn}(\mathcal{C}, \mathcal{V}(\mathbb{1}, -))$ is the bicategory underlying $\mathcal{C}$ (see Definition 3.2.2), so in this case the induced natural transformation may be thought of as an inclusion (see Theorem 3.2.5).

Thus we are compelled to understand monoidal 2-natural transformations of the above form. To that end, let (C, μ, η) denote the corresponding comonoid structure on C from Theorem 3.1.3. By the $\mathcal{C}at$-enriched Yoneda lemma (Lemma 3.1.1), defining a 2-natural transformation $\alpha\colon \mathcal{V}(C,-) \Rightarrow \mathcal{F}$ is the same data as picking an object $c \in \mathcal{F}(C)$. However, not every choice of such an object yields a monoidal 2-natural transformation, since this additionally requires the diagrams

$$\begin{array}{ccc} \mathcal{V}(C,X)\times\mathcal{V}(C,Y) & \xrightarrow{\alpha\times\alpha} & \mathcal{F}(X)\times\mathcal{F}(Y) \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\epsilon} \\ \mathcal{V}(C,X\otimes Y) & \xrightarrow{\alpha} & \mathcal{F}(X\otimes Y) \end{array} \qquad \begin{array}{ccc} & \{*\} & \\ {\scriptstyle v}\swarrow & & \searrow{\scriptstyle u} \\ \mathcal{V}(C,\mathbb{1}) & \xrightarrow{\alpha} & \mathcal{F}(\mathbb{1}) \end{array} \tag{3.1}$$

to commute for all objects $X, Y \in \mathcal{V}$. Observe that the left diagram in diagram (3.1) demands an equality of 2-natural transformations $\mathcal{V}(C,-)\times\mathcal{V}(C,-) \Rightarrow \mathcal{F}(-\otimes-)$, which by Lemma 3.1.2 is equivalent to asking that they agree on the object $(\mathrm{id}_C, \mathrm{id}_C) \in \mathcal{V}(C,C)\times\mathcal{V}(C,C)$. Using the definition of δ, α and the equality $\alpha(C)(\mathrm{id}_C) = c$, this means that the left diagram commutes if and only if

$$\epsilon(C,C)(c,c) = \mathcal{F}(\mu)(c) \in \mathcal{F}(C\otimes C).$$

Similarly, since the $\mathbb{1}$-component of α is

$$\alpha(\mathbb{1})\colon\ \mathcal{V}(C,\mathbb{1}) \xrightarrow{\mathcal{F}} [\mathcal{F}(C), \mathcal{F}(\mathbb{1})] \xrightarrow{\mathrm{eval}_c} \mathcal{F}(\mathbb{1}),$$

[2] In fact, the Yoneda embedding exhibits $[\mathcal{C}, \mathcal{V}]$ as the free monoidal cocompletion of $\mathcal{C}$ [IK86, Thm 5.1].

the right diagram in (3.1) commutes if and only if

$$\mathcal{F}(\eta)(c) = u(*) \in \mathcal{F}(\mathbb{1}).$$

We summarize our result, characterizing monoidal 2-natural transformations $\alpha\colon (\mathcal{V}(C,-),\delta,v) \Rightarrow (\mathcal{F},\epsilon,u)$. This can be seen as a version of the Yoneda lemma for monoidal 2-categories.

Lemma 3.1.5 (Yoneda lemma for monoidal 2-categories). Let $(\mathcal{V},\otimes,\mathbb{1})$ be a small monoidal 2-category and $(\mathcal{F}\colon \mathcal{V} \to \mathcal{C}at, \epsilon, u)$ be a changer. Denote the comonoid in $\mathcal{V}$ corresponding to the represented changer $(\mathcal{V}(C,-),\delta,v)$ by (C,μ,η).
The bijection from Lemma 3.1.1 between 2-natural transformations $\alpha\colon \mathcal{V}(C,-) \Rightarrow \mathcal{F}$ and objects $c \in \mathcal{F}(C)$ restricts to a bijection between monoidal 2-natural transformations $\alpha\colon (\mathcal{V}(C,-),\delta,v) \Rightarrow (\mathcal{F},\epsilon,u)$ and objects $c \in \mathcal{F}(C)$ satisfying

$$\mathcal{F}(\mu)(c) = \epsilon(C,C)(c,c) \in \mathcal{F}(C\otimes C), \qquad \mathcal{F}(\eta)(c) = u(*) \in \mathcal{F}(\mathbb{1}).$$

If the changer $(\mathcal{F}\colon \mathcal{V} \to \mathcal{C}at, \epsilon, u)$ is strong monoidal, then it preserves comonoids, so a comonoid (C,μ,η) in $\mathcal{V}$ induces the comonoid (in $(\mathcal{C}at,\times,\{*\})$)

$$\big(\mathcal{F}(C),\quad \mathcal{F}(C) \xrightarrow{\mathcal{F}(\mu)} \mathcal{F}(C\otimes C) \xrightarrow{\epsilon^{-1}} \mathcal{F}(C)\times\mathcal{F}(C),\;\; \mathcal{F}(C) \xrightarrow{\mathcal{F}(\eta)} \mathcal{F}(\mathbb{1}) \xrightarrow{u^{-1}} \{*\}\big).$$

The requirements on the object $c \in \mathcal{F}(C)$ then precisely state that the corresponding functor $\{*\} \to \mathcal{F}(C)$ is a morphism of comonoids (where $\{*\}$ carries the trivial comonoid structure).
It is well-known that in a cartesian monoidal category, any object carries a unique comonoid structure and any morphism becomes a morphism of comonoids. Consequently, if $\mathcal{F}$ is strong monoidal, 2-natural transformations $\mathcal{V}(C,-) \Rightarrow \mathcal{F}$ are automatically lax monoidal.

Theorem 3.1.6. If $(\mathcal{F}\colon \mathcal{V} \to \mathcal{C}at, \epsilon, u)$ is a strong changer, then lax monoidal 2-natural transformations $\mathcal{V}(C,-) \Rightarrow \mathcal{F}$ correspond to the choice of an object $c \in \mathcal{F}(C)$.

Combining the 2-functoriality of change of enrichment (Theorem 2.6.2) with Lemma 3.1.5 establishes the following result, creating a relationship between change of enrichment along a suitable represented functor and along an arbitrary changer.

Theorem 3.1.7. Let $(\mathcal{V},\otimes,\mathbb{1})$ be a small monoidal 2-category and $(\mathcal{F}\colon \mathcal{V} \to \mathcal{C}at, \epsilon, u)$ be a changer. Let $C \in \mathcal{V}$ be an object carrying a comonoid structure (C,μ,η), which by Theorem 3.1.3 is equivalent to the represented functor $\mathcal{V}(C,-)\colon \mathcal{V} \to \mathcal{C}at$ being equipped with the structure of a changer.
Any object $c \in \mathcal{F}(C)$ satisfying

$$\mathcal{F}(\mu)(c) = \epsilon(C,C)(c,c) \in \mathcal{F}(C\otimes C), \qquad \mathcal{F}(\eta)(c) = u(*) \in \mathcal{F}(\mathbb{1})$$

induces a natural transformation

$$\mathcal{V}\text{-}\mathcal{C}at \underset{\mathrm{ChEn}(-,\mathcal{F})}{\overset{\mathrm{ChEn}(-,\mathcal{V}(C,-))}{\Downarrow}} \mathcal{B}icat.$$

The component $\mathrm{ChEn}(\mathcal{C},\mathcal{V}(C,-)) \to \mathrm{ChEn}(\mathcal{C},\mathcal{F})$ at $\mathcal{C} \in \mathcal{V}\text{-}\mathcal{C}at$ is a strict functor that is the identity on objects and acts via

$$\mathcal{V}(C,\mathcal{C}(A,B)) \xrightarrow{\;\mathcal{F}\;} [\mathcal{F}(C),\mathcal{F}(\mathcal{C}(A,B))] \xrightarrow{\;\mathrm{eval}_c\;} \mathcal{F}(\mathcal{C}(A,B)).$$

3.2 Underlying Bicategory

Any monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$ carries the structure of a comonoid on its unit object $\mathbb{1}$, where the comultiplication $\mu\colon \mathbb{1} \to \mathbb{1} \otimes \mathbb{1}$ is the isomorphism given by the $\mathcal{V}$-associators[3] and the counit $\mathbb{1} \to \mathbb{1}$ is the identity.
If $(\mathcal{V}, \otimes, \mathbb{1})$ is a (small) monoidal 2-category, then by Theorem 3.1.3, this comonoid yields a changer

$$(\mathcal{V}(\mathbb{1}, -)\colon (\mathcal{V}, \otimes, \mathbb{1}) \longrightarrow (\mathcal{C}at, \times, \{*\}), \ \ \epsilon, \ \ u),$$

where the 2-natural transformation $\epsilon\colon \mathcal{V}(\mathbb{1}, -) \times \mathcal{V}(\mathbb{1}, -) \Rightarrow \mathcal{V}(\mathbb{1}, - \otimes -)$ is comprised of the components

$$\epsilon(A, B)\colon \ \ \mathcal{V}(\mathbb{1}, A) \times \mathcal{V}(\mathbb{1}, B) \xrightarrow{\otimes} \mathcal{V}(\mathbb{1} \otimes \mathbb{1}, A \otimes B) \xrightarrow{-\circ\mu} \mathcal{V}(\mathbb{1}, A \otimes B)$$

and the functor $u\colon \{*\} \to \mathcal{V}(\mathbb{1}, \mathbb{1})$ picks the identity $\mathrm{id}_{\mathbb{1}}$.

Example 3.2.1.

1. For $(\mathcal{V}, \otimes, \mathbb{1}) = (\mathcal{C}at, \times, \{*\})$, the functor $\mathcal{V}(\{*\}, -)\colon \mathcal{C}at \to \mathcal{C}at$ is the identity, if we identify functors $\{*\} \to C$ and natural transformations between them by their images.

2. If $(\mathcal{V}, \otimes, \mathbb{1})$ is a monoidal category considered as a monoidal 2-category with only identity 2-cells, then $\mathcal{V}(\mathbb{1}, -)\colon \mathcal{V} \to \mathcal{C}at$ factors over the inclusion functor $(\mathcal{S}et, \times, \{*\}) \hookrightarrow (\mathcal{C}at, \times, \{*\})$:

$$\mathcal{V} \xrightarrow{\mathcal{V}(\mathbb{1}, -)} \mathcal{S}et \hookrightarrow \mathcal{C}at.$$

   For instance, if $(\mathcal{V}, \otimes, \mathbb{1}) = (\mathcal{M}od_R, \otimes_R, R)$ is the category of R-modules over a commutative ring R, then the functor $\mathcal{M}od_R(R, -)\colon \mathcal{M}od_R \to \mathcal{S}et$ is (isomorphic to) the forgetful functor $\mathcal{M}od_R \to \mathcal{S}et$. Similarly, the forgetful functor $\mathcal{T}op \to \mathcal{S}et$ is represented by the singleton topological space $\{*\}$.

Definition 3.2.2. By Theorem 2.5.6, changing the enrichment using the changer $(\mathcal{V}(\mathbb{1}, -), \epsilon, u)$ produces the **underlying bicategory** functor

$$(-)_0\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{B}icat.$$

As the name suggests, this functor assigns a given $\mathcal{V}$-enriched category the bicategory that "underlies" it.

Example 3.2.3.

1. Of course, the bicategory underlying a bicategory is itself; that is, if $(\mathcal{V}, \otimes, \mathbb{1}) = (\mathcal{C}at, \times, \{*\})$, then the functor $(-)_0\colon \mathcal{B}icat \to \mathcal{B}icat$ is (isomorphic to) the identity. Note that this follows immediately from the isomorphism $\mathcal{C}at(\{*\}, -) \cong \mathrm{id}$ and the 2-functoriality of change of enrichment (Theorem 2.6.2).

2. Regarding a monoidal category $(\mathcal{V}, \otimes, \mathbb{1})$ as a monoidal 2-category with only identity 2-cells, the factorization from Example 3.2.1 gives (by functoriality) rise to the factorization

$$(-)_0\colon \mathcal{V}\text{-}\mathcal{C}at \to \mathcal{C}at \hookrightarrow \mathcal{B}icat.$$

   Here we used that the inclusion $\mathcal{S}et \hookrightarrow \mathcal{C}at$ is product-preserving and thus a cartesian changer. The functor $\mathcal{V}\text{-}\mathcal{C}at \to \mathcal{C}at$ is the *underlying category* functor for (ordinary) enriched categories, as it is treated e.g. in [Kel05, Sec 1.3].

The significance of the functor represented by the unit object is demonstrated by the following result.

Lemma 3.2.4. The changer $\mathcal{V}(\mathbb{1}, -)\colon \mathcal{V} \to \mathcal{C}at$ (equipped with the structure from above) is an initial object in the category of changers $(\mathcal{V}, \otimes, \mathbb{1}) \to (\mathcal{C}at, \times, \mathbb{1})$.

[3] In a monoidal category, the $\mathbb{1}$-component of the left and right unitor are equal. This follows from the coherence theorem for monoidal categories or can be deduced directly (see [Eti+16, Cor 2.2.5]).

Proof. Let $(\mathcal{F}: \mathcal{V} \to \mathcal{C}at, \epsilon, u)$ be a changer. By Lemma 3.1.5, monoidal 2-natural transformations

$$\alpha: \mathcal{V}(C, -) \Rightarrow \mathcal{F}$$

correspond to objects $c \in \mathcal{F}(\mathbb{1})$, such that

$$\mathcal{F}(\lambda^{-1}(\mathbb{1}))(c) = \epsilon(\mathbb{1}, \mathbb{1})(c, c) \in \mathcal{F}(\mathbb{1} \otimes \mathbb{1}), \qquad \mathcal{F}(\mathrm{id}_{\mathbb{1}})(c) = u(*) \in \mathcal{F}(\mathbb{1}).$$

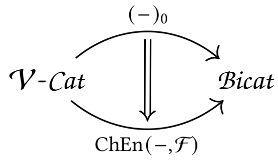
By functoriality of $\mathcal{F}$, the second condition forces $c = u(*)$ and the unitality of $\mathcal{F}$ shows that $\mathcal{F}(\lambda^{-1}(\mathbb{1}))$ is equal to

$$\mathcal{F}(\mathbb{1}) \xrightarrow{\cong} \{*\} \times \mathcal{F}(\mathbb{1}) \xrightarrow{u \times \mathrm{id}} \mathcal{F}(\mathbb{1}) \times \mathcal{F}(\mathbb{1}) \xrightarrow{\epsilon} \mathcal{F}(\mathbb{1} \otimes \mathbb{1}).$$

We finish the proof by noting that this composition carries $c = u(*)$ to $\epsilon(\mathbb{1}, \mathbb{1})(c, c)$, as desired. □

Because of this universal property, the underlying bicategory is the "smallest" bicategory, that can be obtained from a given $\mathcal{V}$-enriched category using a changer $\mathcal{V} \to \mathcal{C}at$. This is made precise by the following theorem, which is a version of Theorem 3.1.7.

Theorem 3.2.5. Let $(\mathcal{V}, \otimes, \mathbb{1})$ be a small monoidal 2-category and $(\mathcal{F}: \mathcal{V} \to \mathcal{C}at, \epsilon, u)$ be a changer. Denote the unique object in the image of u by $u_* := u(*) \in \mathcal{F}(\mathbb{1})$. Then there exists a natural transformation



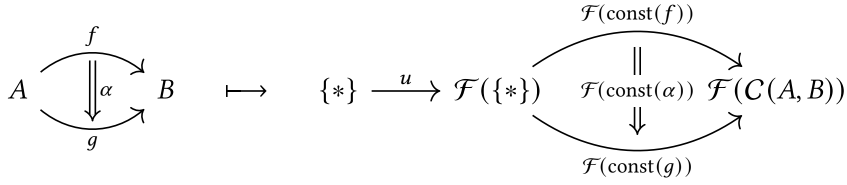
whose components $C_0 \to \mathrm{ChEn}(C, \mathcal{F})$ are strict functors that are the identity on objects and are determined by

$$\mathcal{V}(\mathbb{1}, C(A, B)) \xrightarrow{\mathcal{F}} [\mathcal{F}(\mathbb{1}), \mathcal{F}(C(A, B))] \xrightarrow{\mathrm{eval}_{u_*}} \mathcal{F}(C(A, B)).$$

Proof. By Lemma 3.2.4, there exists a unique monoidal 2-natural transformation $\mathcal{V}(\mathbb{1}, -) \Rightarrow \mathcal{F}$ induced by $u_* \in \mathcal{F}(\mathbb{1})$, so the 2-functoriality of change of enrichment (Theorem 2.6.2) yields the assertion. □

This canonical natural transformation describes the way in which the underlying bicategory C_0 actually underlies the bicategory $C_{\mathcal{F}}$. Changing perspectives, this also allows us to think of $C_{\mathcal{F}}$ as an "extension" of C_0.

Example 3.2.6. Let us spell this out in the case of the cartesian monoidal category $(\mathcal{C}at, \times, \{*\})$. As mentioned in Example 3.2.1, in this case $(-)_0$ is the identity functor $\mathcal{C}at \to \mathcal{C}at$, assuming we identify functors $\{*\} \to C$ and natural transformations between them by their images.
Thus for any changer $(\mathcal{F}: \mathcal{C}at \to \mathcal{C}at, \epsilon, u)$, Theorem 3.2.5 produces a natural transformation $\mathrm{id} \Rightarrow \mathrm{ChEn}(-, \mathcal{F})$. The component $C \to \mathrm{ChEn}(C, \mathcal{F})$ at a fixed bicategory C acts via



for objects $A, B \in C$.

3.3 Represented Changers for cartesian monoidal 2-Categories

It is well-known that any object $C \in \mathcal{V}$ in a cartesian monoidal category $(\mathcal{V}, \times, \{*\})$ has the unique structure of a comonoid, where the comultiplication is given by the diagonal morphism $C \to C \times C$ and the counit is the unique morphism $C \to \{*\}$ into the terminal object $\{*\}$.

By Theorem 3.1.3, this implies that in a cartesian monoidal 2-category, every represented functor

$$\mathcal{V}(C, -)\colon \mathcal{V} \to \mathcal{C}at$$

becomes a changer in a unique way. The components of the corresponding 2-natural transformation $\epsilon\colon \mathcal{V}(C,-) \times \mathcal{V}(C,-) \Rightarrow \mathcal{V}(C, - \times -)$ are given by the functor induced by the universal property of the product

$$\epsilon(A,B)\colon \mathcal{V}(C,A) \times \mathcal{V}(C,B) \to \mathcal{V}(C, A \times B).$$

This reflects the fact that $\mathcal{V}(C,-)$, being a represented functor, preserves limits and thus products. In particular, $\mathcal{V}(C,-)$ is a cartesian changer.

Applying this to the cartesian monoidal category $(\mathcal{C}at, \times, \{*\})$ yields multiple interesting examples.

Example 3.3.1. Fix an arbitrary category C. The internal hom of $\mathcal{C}at$ gives the changer

$$\mathcal{C}at(C,-)\colon \mathcal{C}at \to \mathcal{C}at,\ \mathcal{D} \mapsto [C, \mathcal{D}]$$

transforming a category $\mathcal{D}$ into the category of functors $C \to \mathcal{D}$ and acting by postcomposition on the functor category $[\mathcal{D}, \mathcal{E}]$.

Changing the enrichment (see Construction 2.4.1) along this changer transforms bicategories into bicategories. The special case of monoidal categories was covered in Example 2.4.3. We illustrate this further by mentioning some particular choices of C:

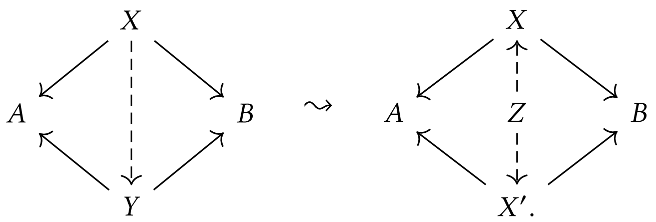
1. Let $C = \bullet \to \bullet$ be the interval category, so that $[C, \mathcal{D}] = \mathrm{Mor}(\mathcal{D})$ is the arrow category. Then change of enrichment implies that any bicategory $\mathcal{D}$ can be extended to a bicategory on the same objects such that $\mathcal{D}(X,Y)$ becomes $\mathrm{Mor}(\mathcal{D}(X,Y))$.
In particular, if $\mathcal{D}$ is a monoidal category, then $\mathrm{Mor}(\mathcal{D})$ is also monoidal.

2. If C is a discrete category, then the functor category $[C, \mathcal{D}]$ is isomorphic to the product category $\prod_{V \in C} \mathcal{D}$. Therefore, a bicategory $\mathcal{D}$ can be extended to another bicategory by replacing the category $\mathcal{D}(X,Y)$ with $\prod_{V \in C} \mathcal{D}(X,Y)$.

3. Let $C = \Delta^{\mathrm{op}}$ be the opposite of the simplex category Δ. Then $[C, \mathcal{D}]$ is the category of simplicial objects in $\mathcal{D}$. It follows that replacing $\mathcal{D}(X,Y)$ in a bicategory $\mathcal{D}$ by the corresponding simplicial object gives rise to another bicategory.
For example, this produces the cartesian monoidal category of simplicial sets $(s\mathcal{S}et, \times, \{*\})$ from $(\mathcal{S}et, \times, \{*\})$.

4 Application to the Bicategory of Spans

As an application of changing the enrichment for categories enriched over monoidal 2-categories, we explain how to construct an extended bicategory of spans $\mathcal{S}pan_{ext}(C)$ from the ordinary bicategory of spans $\mathcal{S}pan(C)$. $\mathcal{S}pan_{ext}(C)$ has the same objects and 1-cells, but its 2-cells are equivalence classes of spans instead of the usual morphisms in C:



Since we essentially want to replace the 2-cells of $\mathcal{S}pan(C)$, the idea of using change of enrichment presents itself.

The chapter starts by introducing the ordinary category of spans $\mathcal{S}pan_1(C)$ and observing its functoriality, which yields the changer $\mathcal{S}pan_1$ (see Definition 4.1.4) implementing the replacement of 2-cells described above. Then we define the bicategory of spans and mention some of its properties. We finish the chapter with the aforementioned construction of the extended bicategory of spans and explain how some of the properties of the bicategory of spans "formally" translate to the extended bicategory of spans.

4.1 The Category of Spans

As a first step towards the bicategory of spans, we motivate and define the ordinary category of spans.

Let C be a category. By definition, the morphisms are "asymmetric" in the sense that they have an explicitly defined domain and codomain. A natural idea is to "symmetrize" the category C by keeping its objects and replacing its ordinary class of morphisms by *spans* (also called *correspondences*); i.e. pairs of morphisms $f: X \to B$, $g: X \to B$, written as

$$A \xleftarrow{f} X \xrightarrow{g} B.$$

Note that a span with $f = \mathrm{id}_A$ amount to a morphism $g: A \to B$ in C and a span with $g = \mathrm{id}_B$ is essentially just a morphism $f: B \to A$. In this sense, spans really eliminate the asymmetry between domain and codomain of a morphism.

Furthermore, whenever the ambient category C has products, the data of a span $A \xleftarrow{f} X \xrightarrow{g} B$ is the same as a morphism $X \to A \times B$ into the product, allowing us to view spans as "generalized relations", see Example 4.1.2.

Assuming C has pullbacks, spans can then be composed by taking pullbacks, giving a category of spans.

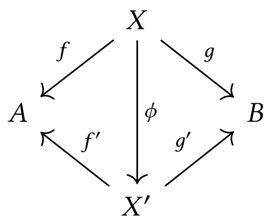
Definition 4.1.1. Let C be a category with pullbacks. The corresponding **category of spans** $\mathcal{S}pan_1(C)$ has as objects the objects of C. A morphism $A \to B$ is an equivalence class of *spans*

$$A \xleftarrow{f} X \xrightarrow{g} B,$$

where two spans

$$A \xleftarrow{f} X \xrightarrow{g} B, \qquad A \xleftarrow{f'} X' \xrightarrow{g'} B$$

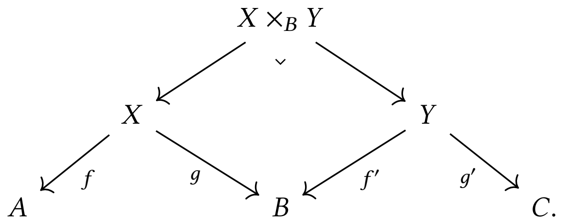
are identified if there exists an isomorphism $\phi\colon X \to X'$ making the diagram



commute. The composition of two spans

$$A \xleftarrow{f} X \xrightarrow{g} B, \qquad\qquad B \xleftarrow{f'} Y \xrightarrow{g'} C$$

is given by the pullback



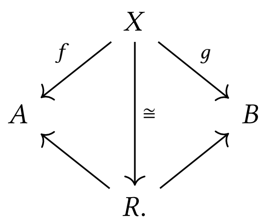
Technically, for this to be a proper category, we have to choose pullbacks (using the axiom of choice), so that composition of functions actually constitutes a function $\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$. The identity $A \to A$ is the span

$$A \xleftarrow{\mathrm{id}_A} A \xrightarrow{\mathrm{id}_A} A.$$

The equivalence classes are needed so that composition is associative and the above span is an identity.

Example 4.1.2.

1. Consider the category of sets $\mathcal{Set}$ consisting of the sets as objects and functions between them as morphisms. The objects in its category of spans $\mathcal{Span}_1(\mathcal{Set})$ are just sets and the morphisms are equivalence classes of spans consisting of two functions $A \xleftarrow{f} X \xrightarrow{g} B$. If f or g is injective, then the induced morphism $X \to A \times B$ is injective and thus constitutes an isomorphism onto its image $R \subset A \times B$, which is simply a relation. Therefore, as a morphism in $\mathcal{Span}_1(\mathcal{Set})$, $A \xleftarrow{f} X \xrightarrow{g} B$ represents the same equivalence class as the relation $R \subset A \times B$:



   By definition, the composition of two relations $R \subset A \times B$ and $S \subset B \times C$ is the relation

$$S \circ R = \{(a, c) \in A \times C : \exists\, b \in B : (a, b) \in R, (b, c) \in S\} \subset A \times C.$$

   It follows that $\mathcal{Span}_1(\mathcal{Set})$ contains the well-known *category of relations $\mathcal{Rel}$* as a subcategory.

2. A more algebraic version of the previous example is obtained by replacing $\mathcal{Set}$ with the category $\mathcal{Mod}_R$ of left R-modules over a fixed ring R. The same argument as in the previous example then leads to the *subcategory of additive relations $\mathcal{AddRel}$*, whose objects are R-modules and whose morphisms are additive relations. This means that a morphism $K\colon M \to N$ is a submodule of the direct sum $M \oplus N$.

3. Similarly, one can fix a group G and consider left G-sets. In this case, a morphism $S\colon X \to Y$ in the corresponding subcategory is a G-invariant subsets of $X \times Y$ (with group action $g.(x,y) \coloneqq (g.x, g.y)$).

Definition 4.1.3. Let $\mathcal{Cat}$ denote the 2-category of (small) categories. The 2-subcategory $\mathcal{Cat}_{pb} \subset \mathcal{Cat}$ consists of the (small) categories with pullbacks as objects and the functors that preserve pullbacks as 1-cells. Its 2-cells are those natural transformations $\alpha\colon F \Rightarrow G$ whose naturality squares

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha(A)} & G(A) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow[\alpha(B)]{} & G(B) \end{array}$$

are pullbacks for all morphisms $f\colon A \to B$. Lemma 4.2.5 guarantees that this actually forms a 2-subcategory in that it is closed under composition of 2-cells.

Definition 4.1.4. We observe that the category of spans construction extends to a strict functor

$$\mathcal{Span}_1\colon \mathcal{Cat}_{pb} \to \mathcal{Cat},\ \mathcal{C} \mapsto \mathcal{Span}_1(\mathcal{C}).$$

It maps a pullback-preserving functor $F\colon \mathcal{C} \to \mathcal{D}$ to the functor $\mathcal{Span}_1(F)\colon \mathcal{Span}_1(\mathcal{C}) \to \mathcal{Span}_1(\mathcal{D})$ which applies F to each span. A natural transformation $\alpha\colon F \Rightarrow G$ whose naturality squares are pullbacks yields the natural transformation $\mathcal{Span}_1(\alpha)\colon \mathcal{Span}_1(F) \to \mathcal{Span}_1(G)$ with components

$$F(A) \xleftarrow{\mathrm{id}_{F(A)}} F(A) \xrightarrow{\alpha(A)} G(A).$$

That $\mathcal{Span}_1(\alpha)$ indeed defines a natural transformation follows from the assumption that the naturality squares of α are pullback squares.

Remark 4.1.5. For any category $\mathcal{C}$ with pullbacks, there is an inclusion functor

$$\iota\colon \mathcal{C} \hookrightarrow \mathcal{Span}_1(\mathcal{C}),$$

which is the identity on objects and maps a morphism $f\colon A \to B$ to the span[1] $A \xleftarrow{\mathrm{id}_A} A \xrightarrow{f} B$. ○

Note that we can also view $\mathcal{Span}_1$ as a functor into the arrow category $\mathrm{Mor}(\mathcal{Cat}) \coloneqq [\bullet \to \bullet, \mathcal{Cat}]$:

$$\mathcal{Span}_1\colon \mathcal{Cat}_{pb} \to \mathrm{Mor}(\mathcal{Cat}),\ \mathcal{C} \mapsto \big(\iota\colon \mathcal{C} \hookrightarrow \mathcal{Span}_1(\mathcal{C})\big),\ F \mapsto \big(F, \mathcal{Span}_1(F)\big).$$

The symmetry of the category of spans $\mathcal{Span}_1(\mathcal{C})$ manifests itself in the fact that it is a dagger category.

Lemma 4.1.6. For a category with pullbacks $\mathcal{C}$, its category of spans $\mathcal{Span}_1(\mathcal{C})$ is a dagger category when equipped with the involution $\dagger\colon \mathrm{Hom}_{\mathcal{Span}_1(\mathcal{C})}(A,B) \to \mathrm{Hom}_{\mathcal{Span}_1(\mathcal{C})}(B,A)$ that "mirrors" a span:

$$\dagger\colon \Big(A \xleftarrow{f} X \xrightarrow{g} B\Big) \longmapsto \Big(B \xleftarrow{g} X \xrightarrow{f} A\Big).$$

Example 4.1.7. The category of relations $\mathcal{Rel}$ from Example 4.1.2 inherits this dagger category structure, mapping a relation $R \subset A \times B$ to its *converse relation*

$$R^\dagger \coloneqq \{(b,a) \in B \times A : (a,b) \in R\}.$$

Of course, the same is true for the category of additive relations $\mathcal{AddRel}$. In fact, this symmetry of additive relations has already been observed in [Mac61], making it one of the first appearances of dagger categories (though the terminology "dagger category" was not yet used).

[1]Technically, this is not a span but an equivalence class of spans. For convenience we suppress this from the notation.

4.2 The Bicategory of Spans

The fact that the morphisms in $\mathcal{Span}_1(\mathcal{C})$ as defined in Definition 4.1.1 are equivalence classes of spans instead of actual spans suggests that spans carry additional structure that is not fully captured by the notion of a category.

This becomes especially obvious when noting that a span in $\mathcal{C}$ is equivalently described as a functor from the preorder category $\bullet \leftarrow \bullet \rightarrow \bullet$ to $\mathcal{C}$. A span from A to B is then a functor $(\bullet \leftarrow \bullet \rightarrow \bullet) \rightarrow \mathcal{C}$, mapping the leftmost point to A and the rightmost point to B. This may also be described as the limit of the diagram

$$\bullet \xrightarrow{A} \mathcal{C} \xleftarrow{C} \mathrm{Mor}(\mathcal{C}) \xrightarrow{D} \mathcal{C} \xleftarrow{D} \mathrm{Mor}(\mathcal{C}) \xrightarrow{C} \mathcal{C} \xleftarrow{B} \bullet$$

in $\mathcal{Cat}$, where $\mathrm{Mor}(\mathcal{C}) = [\bullet \rightarrow \bullet, \mathcal{C}]$ is the arrow category and D and C denote the domain and codomain functor, respectively. Indeed, this category is just the category of cones of the functor

$$H\colon \{\bullet, *\} \rightarrow \mathcal{C}, \ \bullet \mapsto A, \ * \mapsto B$$

and the terminal object of this category (if existent) is by definition the product $A \times B$.

This description as functors makes it apparent that the collection of spans from A to B is really a (functor) category.

Definition 4.2.1. Let $\mathcal{C}$ be a category and $A, B \in \mathcal{C}$ be two objects. The spans from A to B form a category $\mathcal{Span}(\mathcal{C})(A, B)$ (also denoted by just $\mathcal{Span}(A, B)$). Its objects are spans from A to B

$$A \xleftarrow{f} X \xrightarrow{g} B$$

and a morphism in $\mathcal{Span}(A, B)$ is a morphism ϕ in $\mathcal{C}$ making the diagram

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{f} & & \searrow^{g} & \\ A & & \downarrow{\scriptstyle \phi} & & B \\ & \nwarrow_{f'} & & \nearrow_{g'} & \\ & & X' & & \end{array}$$

commute.

Remark 4.2.2. It is clear that this construction is also functorial in that for any functor $F\colon \mathcal{C} \rightarrow \mathcal{D}$ and objects $A, B \in \mathcal{C}$, we have a functor $\mathcal{Span}(\mathcal{C})(A, B) \rightarrow \mathcal{Span}(\mathcal{D})(F(A), F(B))$, which simply applies F to each span and the morphisms between them.
Viewing the category $\mathcal{Span}(\mathcal{C})(A, B)$ as the category of cones of H (from above), this just amounts to postcomposing H by F. ◯

Example 4.2.3. As mentioned in Example 4.1.2, any relation $R \subset A \times B$ gives rise to a span $A \longleftarrow R \longrightarrow B$. For two relations R and R', a morphism $\phi\colon R \rightarrow R'$ is just an inclusion and the universal property of the product states that $A \times B$ is the largest relation.

Remark 4.2.4. Viewing spans in $\mathcal{C}$ as functors $(\bullet \leftarrow \bullet \rightarrow \bullet) \rightarrow \mathcal{C}$, one may endow them with the structure of a category by allowing arbitrary natural transformations between them; i.e. by studying commutative diagrams

$$\begin{array}{ccccc} A & \longleftarrow & X & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longleftarrow & X' & \longrightarrow & B'. \end{array}$$

Then one can view spans $A \xleftarrow{f} X \xrightarrow{g} B$ as another kind of morphism $A \to B$ (apart from the ordinary morphisms $A \to B$ in C), leading to the weak double category of spans, see e.g. [GP17, Sec 5]. ○

We will now impose the structure of a bicategory on spans, essentially by combining Definition 4.1.1 and Definition 4.2.1. Making this precise requires some well-known properties of pullbacks, as can be found in [Mac98, Exercise III.4.8].

Lemma 4.2.5 (Pullback Lemma). Let C be a category with pullbacks.

1. The pullback along the identity is the identity; i.e. the following diagram is a pullback

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ \big\downarrow f & \lrcorner & \big\downarrow f \\ A & \xrightarrow{\mathrm{id}_A} & A. \end{array}$$

2. Consider a commutative diagram

$$\begin{array}{ccccc} \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\ \big\downarrow & & \big\downarrow & \lrcorner & \big\downarrow \\ \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \end{array}$$

in C and assume that the right square is a pullback. Then the left square is a pullback if and only if the outside rectangle is a pullback.

3. In particular, for the iterated pullback $P \coloneqq (X \times_B Y) \times_Y (Y \times_C Z)$

$$\begin{array}{ccccccc} & & & P & & & \\ & & \swarrow & \vee & \searrow & & \\ & X \times_B Y & & & & Y \times_C Z & \\ \swarrow & \vee & \searrow & & \swarrow & \vee & \searrow \\ X & & & Y & & & Z \\ \swarrow \quad \searrow & & \swarrow & & \searrow & & \swarrow \quad \searrow \\ A \qquad B & & & & & & C \qquad D, \end{array}$$

the left composite rectangle (spanned by the objects P, $Y \times_C Z$, B and X) is a pullback and the same is true for the right composite triangle. Therefore, we obtain an isomorphism of spans

$$X \times_B (Y \times_C Z) \xrightarrow{\cong} P \xrightarrow{\cong} (X \times_B Y) \times_C Z.$$

For example, in $\mathcal{S}et$ with the usual choice of pullbacks, the first part describes the isomorphism

$$A \times_A X = \{(a, x) \in A \times X : f(x) = a\} \to X, \ (a, x) \mapsto x.$$

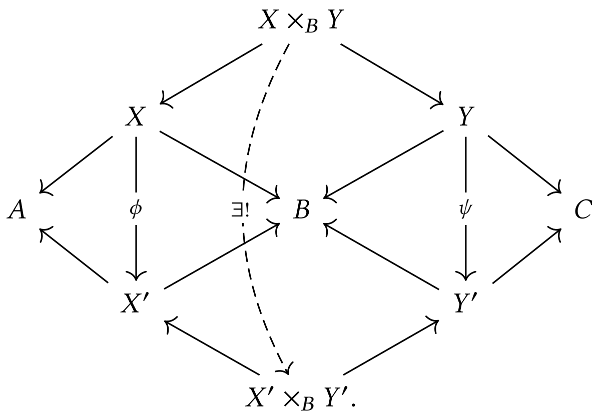
The lemma asserts that taking pullbacks preserves identities and is associative up to isomorphism. Since pullbacks are defined via universal property this is the best we can hope for. These isomorphisms describe the weak associativity and unity in our bicategory of spans, which we can now define.

Definition 4.2.6. Let C be a category with pullbacks. The corresponding **bicategory of spans** $\mathcal{S}pan(C)$ (which for brevity will also be denoted by $\mathcal{S}pan$) consists of the following data:

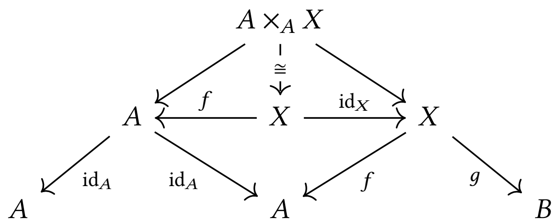
1. The objects are the objects of C: $\mathrm{Ob}(\mathcal{S}pan) = \mathrm{Ob}(C)$.

2. The category $\mathcal{S}pan(A, B)$ is the category from Definition 4.2.1.

3. The identity 1-cell of $A \in \mathcal{C}$ is the span $A \xleftarrow{\mathrm{id}_A} A \xrightarrow{\mathrm{id}_A} A$.
4. The horizontal composition of spans is given by pullback as in Definition 4.1.1 (choosing a pullback for each cospan).
5. The horizontal composition of 2-cells is defined by the universal property of the pullback $X' \times_B Y'$:



6. The left and right unitor are given by the canonical isomorphism from the chosen pullback $A \times_A X$ to X (which exists by Lemma 4.2.5). For instance, for the right unitor ρ, the isomorphism $A \times_A X \to X$

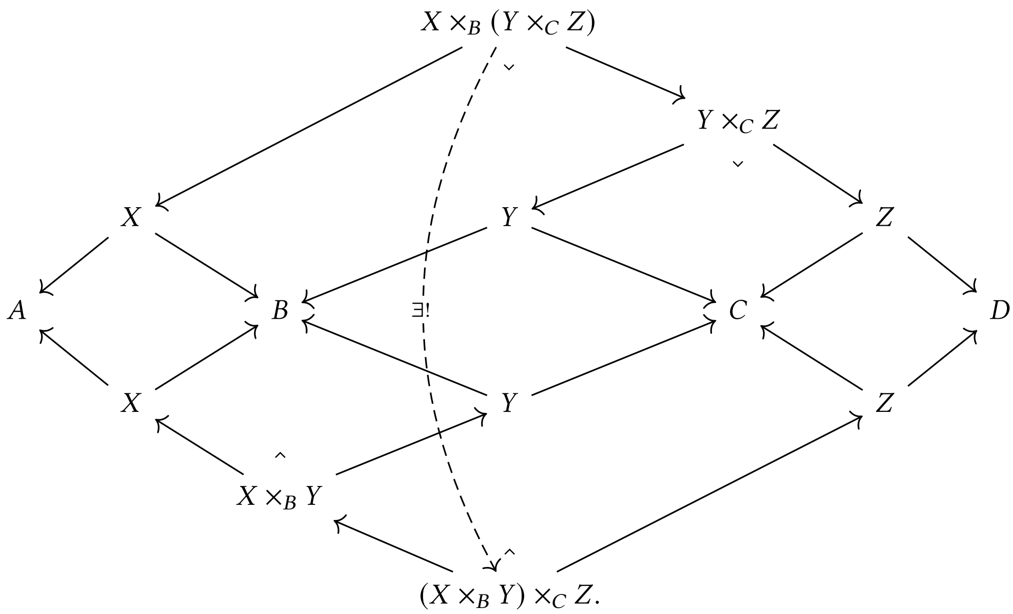


is the component of the natural isomorphism ρ, constituting an isomorphism

$$\Big(A \longleftarrow A \times_A X \longrightarrow X \xrightarrow{g} B \Big) \longrightarrow \Big(A \xleftarrow{f} X \xrightarrow{g} B \Big)$$

in $\mathcal{S}pan(A, B)$.

7. The components of the associator are given by the isomorphism from Lemma 4.2.5:



By choosing the pullback of cospans of the form $A \xrightarrow{\mathrm{id}_A} A \xleftarrow{f} X$ to be $A \xleftarrow{f} X \xrightarrow{\mathrm{id}_X} X$, one can make the right unitor (and analogously the left unitor) the identity.

Of course, because composition is defined using pullbacks and thus is only unique up to isomorphism, it is unreasonable to expect composition of spans to be strictly associative. However, it is still associative up to canonical isomorphism, which is captured by its bicategory structure. It is also possible to construct a version of this bicategory as a Segal object in $\mathcal{C}at$, as is demonstrated in [Ste20, Sec 3]. Further details on the $(\infty,2)$-categorical version of this bicategory can be found in [DK19, Cha 10] and [GR19, Cha 7].

Note that the category $\mathcal{S}pan_1$ from Definition 4.1.1 is the truncation of $\mathcal{S}pan$ in that their objects are the same but the morphisms in $\mathcal{S}pan_1$ are given by isomorphism classes of 1-cells in $\mathcal{S}pan$.

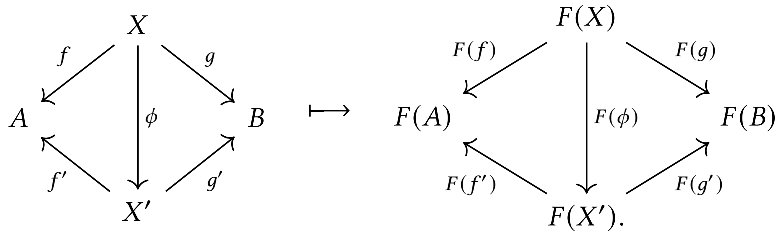
An important property of the span construction $\mathcal{S}pan$ is that it is functorial in the sense that a pullback-preserving functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a functor $\mathcal{S}pan(\mathcal{C}) \to \mathcal{S}pan(\mathcal{D})$. This generalizes the functoriality of its truncation $\mathcal{S}pan_1$, which we observed in Definition 4.1.4.

Theorem 4.2.7 ([Joh+21, Prop 4.1.24])**.** The span construction of Definition 4.2.6 extends to a functor

$$\mathcal{S}pan\colon \mathcal{C}at_{pb} \to \mathcal{B}icat,\ \mathcal{C} \mapsto \mathcal{S}pan(\mathcal{C}),$$

mapping a pullback-preserving functor $F\colon \mathcal{C} \to \mathcal{D}$ to the functor $\mathcal{S}pan(\mathcal{C}) \to \mathcal{S}pan(\mathcal{D})$ that "applies F to everything". More precisely, it applies F on objects and is given by Remark 4.2.2 on hom-categories:

$$\mathcal{S}pan(\mathcal{C})(A,B) \longrightarrow \mathcal{S}pan(\mathcal{D})(F(A),F(B)),$$



[CKS84, Thm 4] identifies the image of this functor (up to equivalence), giving a characterization of the bicategories which arise as the span bicategory of some category with pullbacks.

Intuitively, we may want to view the bicategory of spans $\mathcal{S}pan(\mathcal{C})$ as a "symmetrized extension" of $\mathcal{C}$. In particular, it is beneficial to consider $\mathcal{C}$ as a subcategory of $\mathcal{S}pan(\mathcal{C})$ by identifying a morphism $f\colon A \to B$ with the span $A \xleftarrow{\mathrm{id}_A} A \xrightarrow{f} B$. This idea is captured by the following inclusion functor.

Remark 4.2.8. Analogously to Remark 4.1.5, there is an inclusion 2-functor

$$\iota\colon \mathcal{C} \hookrightarrow \mathcal{S}pan(\mathcal{C}),\ \ A \mapsto A,\ \ (f\colon A \to B) \mapsto \left(A \xleftarrow{\mathrm{id}_A} A \xrightarrow{f} B\right).$$

Here we view $\mathcal{C}$ as a 2-category as described in Example 2.2.2.
Due to the symmetric nature of $\mathcal{S}pan$, there also exists an inclusion functor for the opposite category

$$\mathcal{C}^{\mathrm{op}} \hookrightarrow \mathcal{S}pan(\mathcal{C}),\ \ A \mapsto A,\ \ (f\colon A \to B) \mapsto \left(B \xleftarrow{f} A \xrightarrow{\mathrm{id}_A} A\right).$$ ○

Just like an ordinary category $\mathcal{C}$ has an opposite category $\mathcal{C}^{\mathrm{op}}$, any bicategory admits an opposite bicategory.[2]

Definition 4.2.9. The **opposite bicategory** $\mathcal{C}^{\mathrm{op}}$ of a bicategory $\mathcal{C}$ consists of the same objects as $\mathcal{C}$, the same 2-cells and the direction of the 1-cells reversed; i.e.

$$\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) \coloneqq \mathrm{Ob}(\mathcal{C}), \qquad \mathcal{C}^{\mathrm{op}}(X,Y) \coloneqq \mathcal{C}(Y,X).$$

[2] In fact, a bicategory has additional notions of symmetry (e.g. by reversing the 2-cells), but this will not be relevant for us.

The dagger category structure on the category of spans $\mathcal{Span}_1(C)$ from Lemma 4.1.6 translates to the following result for the bicategory of spans $\mathcal{Span}(C)$.

Lemma 4.2.10. Let C be a category with pullbacks. Then its bicategory of spans $\mathcal{Span}(C)$ admits a functor

$$\mathcal{Span}(C) \to \mathcal{Span}(C)^{\mathrm{op}}$$

acting on objects and 2-cells as the identity and on 1-cells by "mirroring" them:

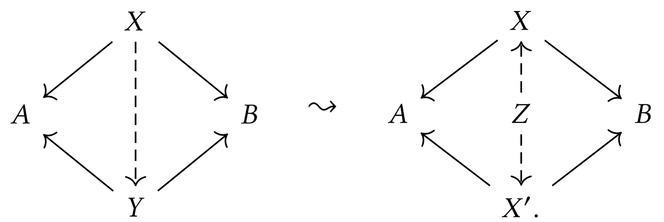
$$\left(A \xleftarrow{f} X \xrightarrow{g} B \right) \quad \longmapsto \quad \left(B \xleftarrow{g} X \xrightarrow{f} A \right).$$

This functor is an involution; in particular $\mathcal{Span}(C) \cong \mathcal{Span}(C)^{\mathrm{op}}$.

A bicategory with this property is occasionally called a *symmetric bicategory* (e.g. in [MS06, Def 16.2.1]). The inclusion functor $C^{\mathrm{op}} \hookrightarrow \mathcal{Span}(C)$ from Remark 4.2.8 is the composition of the inclusion functor $C^{\mathrm{op}} \hookrightarrow \mathcal{Span}(C)^{\mathrm{op}}$ with this isomorphism $\mathcal{Span}(C)^{\mathrm{op}} \cong \mathcal{Span}(C)$.

4.3 Construction of the extended Bicategory of Spans

We can finally construct the extended bicategory of spans from the ordinary bicategory of spans, as defined in Definition 4.2.6. As mentioned, the idea is to formally replace the 2-cells in the bicategory of spans by equivalence classes of spans:



This is achieved by using the strict functor

$$\mathcal{Span}_1 \colon \mathcal{Cat}_{pb} \to \mathcal{Cat},\ C \mapsto \mathcal{Span}_1(C)$$

from Definition 4.1.4 in order to change the enrichment of $\mathcal{Span}$.

Lemma 4.3.1. The 2-category $\mathcal{Cat}_{pb}$ (from Definition 4.1.3) is a monoidal 2-subcategory of $(\mathcal{Cat}, \times, \{*\})$.

Proof. Because the terminal category $\{*\}$ has pullbacks, one has to check that products of categories, functors and natural transformations in $\mathcal{Cat}_{pb}$ lie again in $\mathcal{Cat}_{pb}$. But this is clear as a pullback in the product category $C \times \mathcal{D}$ is the same as a pullback in C and a pullback in $\mathcal{D}$[3]. □

Since the components of the canonical 2-natural transformation

$$\mathcal{Span}_1\left(\prod_{i \in I} - \right) \Rightarrow \prod_{i \in I} \mathcal{Span}_1(-)$$

are isomorphisms, $\mathcal{Span}_1$ preserves products.

Lemma 4.3.2. The 2-functor $\mathcal{Span}_1 \colon \mathcal{Cat}_{pb} \to \mathcal{Cat}$ preserves products and thus becomes a cartesian changer.

We give a special name to changing the enrichment w.r.t. this functor.

[3]The argument (when replacing binary products by arbitrary ones) even shows that $\mathcal{Cat}_{pb}$ is closed under arbitrary products (not just finite ones).

Definition 4.3.3. Changing the enrichment using $\mathcal{S}pan_1$ yields the **spanification functor**

$$\mathrm{Spanify}\colon \mathcal{C}at_{pb}\text{-}\mathcal{C}at \longrightarrow \mathcal{B}icat.$$

By Example 2.5.7, we may view the category of $(\mathcal{C}at_{pb}, \times, \{*\})$-enriched categories $\mathcal{C}at_{pb}\text{-}\mathcal{C}at$ as a subcategory of $\mathcal{B}icat$ via the change of enrichment functor $\mathrm{ChEn}(-,\iota)\colon \mathcal{C}at_{pb}\text{-}\mathcal{C}at \longrightarrow \mathcal{B}icat$ induced by the inclusion $\iota\colon \mathcal{C}at_{pb} \hookrightarrow \mathcal{C}at$. Because we want to apply the spanification functor $\mathrm{Spanify}\colon \mathcal{C}at_{pb}\text{-}\mathcal{C}at \longrightarrow \mathcal{B}icat$ to the bicategory of spans $\mathcal{S}pan$, we have to check that it can be viewed as a $\mathcal{C}at_{pb}$-enriched category. More precisely, this amounts to the following statement.

Lemma 4.3.4. The bicategory of spans (from Definition 4.2.6) is in the image of the change of enrichment functor $\mathrm{ChEn}(-,\iota)\colon \mathcal{C}at_{pb}\text{-}\mathcal{C}at \longrightarrow \mathcal{B}icat$.

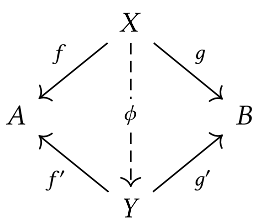
Proof. Example 2.4.2 states that this translates to the following requirements for all $X, Y \in \mathrm{Ob}(C)$:

$$C(X,Y) \in \mathcal{C}at_{pb}, \qquad \mathrm{id}_X^C \in \mathcal{C}at_{pb}, \qquad \bullet \in \mathcal{C}at_{pb}, \qquad \lambda \in \mathcal{C}at_{pb}, \qquad \rho \in \mathcal{C}at_{pb}, \qquad \alpha \in \mathcal{C}at_{pb}.$$

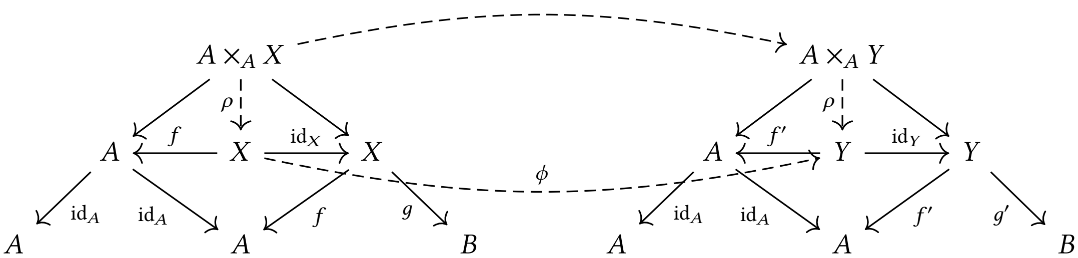
That the hom-categories $C(X,Y)$ have pullbacks is the content of the following Lemma 4.3.5. A couple of diagram chases confirm that $\bullet$, λ, ρ and α of $\mathcal{S}pan$ lie in $\mathcal{C}at_{pb}$. We demonstrate the argument for ρ; that is, we show that the right unitor

$$\rho\colon -\bullet\mathrm{id}_A \overset{\cong}{\Longrightarrow} \mathrm{id}_{\mathcal{S}pan(A,B)}$$

has the property that all its naturality squares are pullback squares.
To that end, let $\phi\colon X \longrightarrow Y$ be a morphism in $\mathcal{S}pan(A,B)$; i.e. a morphism in C making the diagram



commute. By the universal property of the pullback, this induces a morphism $A \times_A X \longrightarrow A \times_A Y$ and we have to show that the dashed rectangle in the diagram



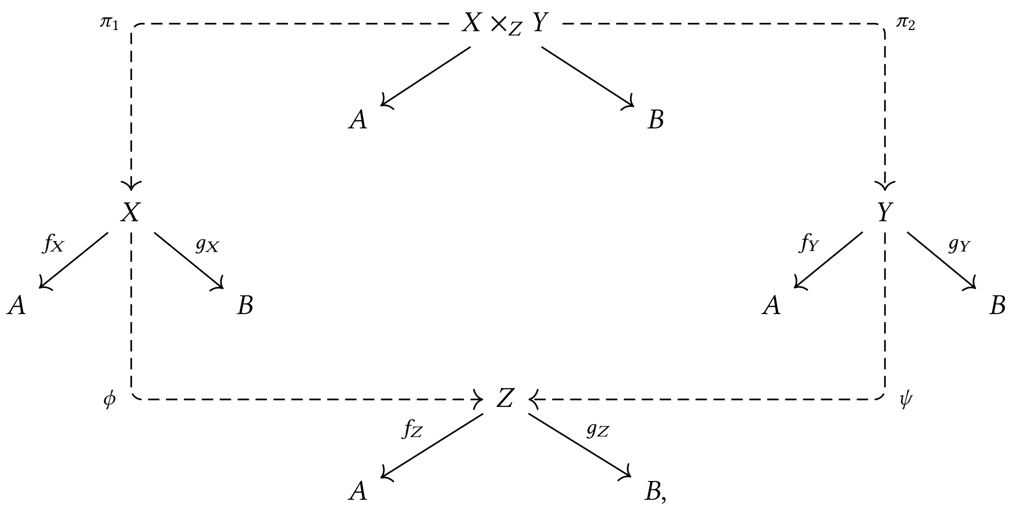
is a pullback square. Indeed, focusing on the subdiagram

$$\begin{array}{ccccc} A\times_A X & \dashrightarrow & A\times_A Y & \longrightarrow & A \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\mathrm{id}_A} \\ X & \overset{\phi}{\dashrightarrow} & Y & \overset{f'}{\longrightarrow} & A, \end{array}$$

we observe that the right square and the outside rectangle are pullback diagrams, which yields the assertion by Lemma 4.2.5. ☐

Lemma 4.3.5. Let C be a category and $A, B \in C$ be two objects. If C has pullbacks, then the same is true for the category of spans $\mathcal{S}pan(A,B)$ from A to B.

Proof. Consider the following diagram



where the lower three spans and the lower two dashed morphisms are given. By definition of the pullback and morphisms in $C(A, B)$, we have

$$f_X \circ \pi_1 = f_Z \circ \phi \circ \pi_1 = f_Z \circ \psi \circ \pi_2 = f_Y \circ \pi_2$$

and similarly

$$g_X \circ \pi_1 = g_Z \circ \phi \circ \pi_1 = g_Z \circ \psi \circ \pi_2 = g_Y \circ \pi_2.$$

These two morphisms form the legs of the span

$$A \longleftarrow X \times_Z Y \longrightarrow B,$$

which together with the canonical morphisms $\pi_1 \colon X \times_Z Y \to X$, $\pi_2 \colon X \times_Z Y \to Y$ is the pullback of ϕ and ψ, as one can check. □

Remark 4.3.6. Note that the category of spans $\mathcal{S}pan(A, B)$ generally does not have all limits, even if C has all limits. For example, consider the category of nonempty sets $\mathcal{S}et \setminus \{\emptyset\}$ and let $A = B = \{0, 1\}$ be a set consisting of two elements. Then the product of the two spans

$$\{0,1\} \hookleftarrow \{0\} \hookrightarrow \{0,1\}, \qquad \{0,1\} \hookleftarrow \{1\} \hookrightarrow \{0,1\}$$

does not exist, because there is no cone for these two spans. ○

Therefore, we may apply the spanification functor Spanify: $\mathcal{C}at_{pb}\text{-}\mathcal{C}at \to \mathcal{B}icat$ to the bicategory of spans $\mathcal{S}pan(C)$ (formally, to its preimage under $\mathrm{ChEn}(-, \iota)$ from Lemma 4.3.4). This produces the following extended bicategory of spans.

Definition 4.3.7. Let C be a category with pullbacks. Its **extended bicategory of spans** $\mathcal{S}pan_{ext}(C)$ (which for brevity is also denoted by $\mathcal{S}pan_{ext}$) is the following bicategory:

1. The objects are the objects of C: $\mathrm{Ob}(\mathcal{S}pan_{ext}) = \mathrm{Ob}(C)$.

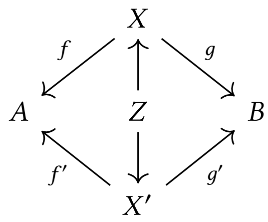
2. The category $\mathcal{S}pan(A, B)$ has as objects spans of the form

$$A \xleftarrow{\;f\;} X \xrightarrow{\;g\;} B.$$

   A morphism

$$\left(A \xleftarrow{\;f\;} X \xrightarrow{\;g\;} B \right) \longrightarrow \left(A \xleftarrow{\;f'\;} X' \xrightarrow{\;g'\;} B \right)$$

is an equivalence class of spans $X \longleftarrow Z \longrightarrow X'$ (w.r.t. the equivalence relation from Definition 4.1.1) making the diagram

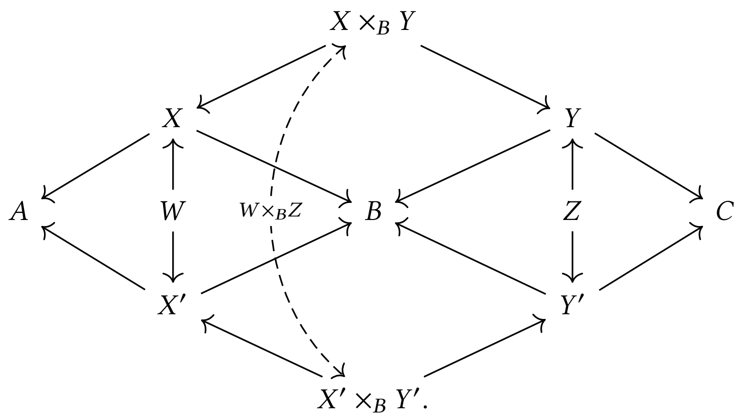


commute. By construction, Z is also equipped with a span $A \longleftarrow Z \longrightarrow B$, but these two morphisms are already determined by the rest of the data. Composition is given by pullback and the identity 2-cell is the span $X \xleftarrow{\mathrm{id}_X} X \xrightarrow{\mathrm{id}_X} X$.

3. The identity 1-cell of $A \in C$ is the span $A \xleftarrow{\mathrm{id}_A} A \xrightarrow{\mathrm{id}_A} A$.

4. The horizontal composition of 1-cells is given by pullback as in Definition 4.1.1.

5. The horizontal composition of 2-cells is



6. The unitors and the associator are obtained by applying $\mathcal{S}pan_1$ to the corresponding data of $\mathcal{S}pan$.

Remark 4.3.8. Roughly, the main difference between the extended bicategory of spans $\mathcal{S}pan_{ext}(C)$ (Definition 4.3.7) and the bicategory of spans $\mathcal{S}pan(C)$ (Definition 4.2.6) is that the former has equivalence classes of spans as 2-cells instead of ordinary morphisms of C. ○

As a consequence of this formal construction of the extended bicategory of spans via change of enrichment, we obtain a number of functoriality results from the functoriality of changing the enrichment. We summarize some of these results in the theorem after the subsequent lemma.

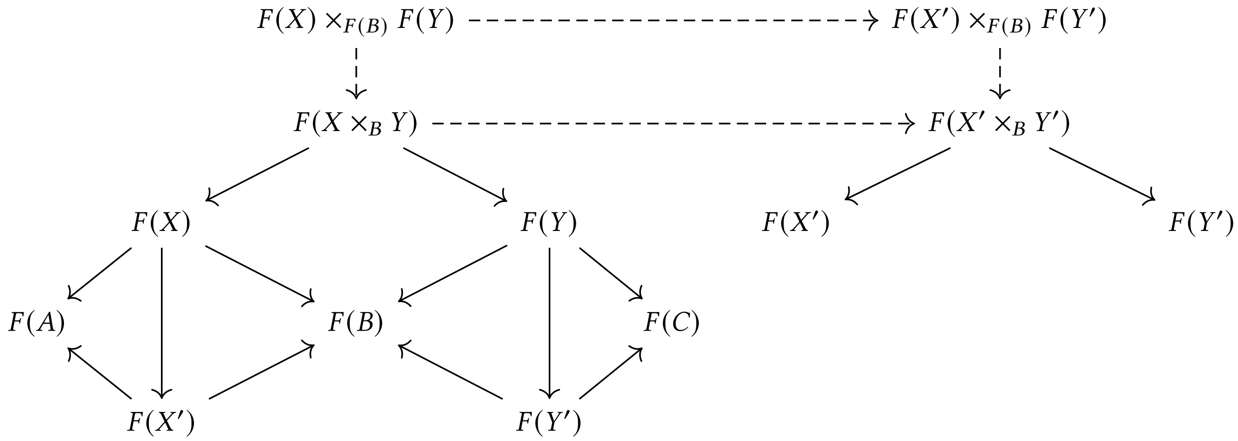
Lemma 4.3.9. The image of the span functor $\mathcal{S}pan\colon \mathcal{C}at_{pb} \to \mathcal{B}icat$ from Theorem 4.2.7 lies in $\mathcal{C}at_{pb}\text{-}\mathcal{C}at$, so we may view $\mathcal{S}pan$ as a functor $\mathcal{S}pan\colon \mathcal{C}at_{pb} \to \mathcal{C}at_{pb}\text{-}\mathcal{C}at$.

Proof. On objects, this is exactly the content of Lemma 4.3.4. For a morphism $F\colon C \to \mathcal{D}$, Example 2.5.7 states that we have to show

$$\mathcal{S}pan(F)(X,Y) \in \mathcal{C}at_{pb} \quad \forall X, Y \in \mathrm{Ob}(C), \qquad \mathcal{S}pan(F)_{\mathrm{id}} \in \mathcal{C}at_{pb}, \qquad \mathcal{S}pan(F)_{\bullet} \in \mathcal{C}at_{pb}.$$

That $\mathcal{S}pan(F)(X,Y)$ preserves pullbacks is a consequence of the concrete form of the pullbacks in the span category, as described in Lemma 4.3.5 and the fact that $\mathcal{S}pan(F)(X,Y)\colon \mathcal{S}pan(X,Y) \to \mathcal{S}pan(F(X),F(Y))$

just applies F to the spans. Because $\mathcal{S}pan(F)_{\mathrm{id}}$ is an identity 2-cell, the second statement is clear. Finally, the third claim amounts to the statement that the dashed rectangle in



is a pullback, where $X \to X'$, $Y \to Y'$ are two morphisms in C. A pair of morphisms

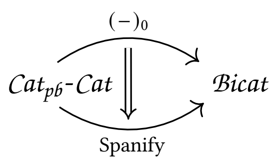
$$T \to F(X') \times_{F(B)} F(Y'), \qquad T \to F(X \times_B Y)$$

making the square into $F(X' \times_B Y')$ commute is, by the universal property of the pullback $F(X') \times_{F(B)} F(Y')$, the same data as two morphisms $T \to F(X')$, $T \to F(Y')$ making the square into $F(B)$ commute. This corresponds exactly to a morphism $T \to F(X') \times_{F(B)} X(Y')$. □

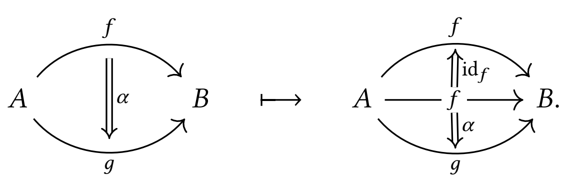
Theorem 4.3.10. The extended bicategory of spans $\mathcal{S}pan_{ext}$ from Definition 4.3.7 has the following properties:

1. The extended bicategory of spans construction $\mathcal{S}pan_{ext}(C)$ is functorial in the input category C; i.e. it extends to a functor
$$\mathcal{S}pan_{ext} \colon \mathcal{C}at_{pb} \to \mathcal{B}icat.$$

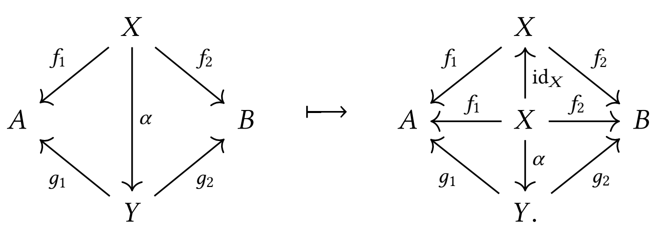
2. There is a natural transformation



from the underlying bicategory functor to the spanification functor. Its components $C \cong C_0 \to \mathrm{Spanify}(C)$ are strict functors that are the identity on objects and are given by Remark 4.1.5 on hom-categories $C(A, B) \to \mathcal{S}pan_1(C(A, B))$:



Precomposing the natural transformation with $\mathcal{S}pan \colon \mathcal{C}at_{pb} \to \mathcal{C}at_{pb}$-$\mathcal{C}at$ yields a natural transformation $\mathcal{S}pan \Rightarrow \mathcal{S}pan_{ext}$ with components (where the morphisms f_1 and f_2 in the middle are usually not drawn as they are determined by the rest of the data)



3. There is an involution functor $\mathcal{S}pan_{ext}(C) \to \mathcal{S}pan_{ext}(C)^{\mathrm{op}}$ fixing objects and 2-cells and "mirroring" the 1-cells:
$$\Big(A \xleftarrow{f} X \xrightarrow{g} B \Big) \quad\longmapsto\quad \Big(B \xleftarrow{g} X \xrightarrow{f} A \Big).$$

Proof. 1. Lemma 4.3.9 shows that the span functor $\mathcal{S}pan\colon \mathcal{C}at_{pb} \to \mathcal{B}icat$ from Theorem 4.2.7 factors as
$$\mathcal{C}at_{pb} \to \mathcal{C}at_{pb}\text{-}\mathcal{C}at \to \mathcal{B}icat$$
and composing the functor $\mathcal{C}at_{pb} \to \mathcal{C}at_{pb}\text{-}\mathcal{C}at$ with the spanification functor Spanify: $\mathcal{C}at_{pb}\text{-}\mathcal{C}at \to \mathcal{B}icat$ yields the desired functor $\mathcal{S}pan_{ext}\colon \mathcal{C}at_{pb} \to \mathcal{B}icat$.

2. This follows by combining Theorem 3.2.5 with the definition of $\mathcal{S}pan_1$ (Definition 4.1.4).

3. By functoriality of changing the enrichment, the functor $\mathcal{S}pan(C) \to \mathcal{S}pan(C)^{\mathrm{op}}$ from Lemma 4.2.10 induces a functor $\mathcal{S}pan_{ext}(C) \to \mathrm{ChEn}\big(\mathcal{S}pan(C)^{\mathrm{op}}, \mathcal{S}pan_1\big)$, which is the desired functor when composed with the canonical isomorphism
$$\mathrm{ChEn}\big(\mathcal{S}pan(C)^{\mathrm{op}}, \mathcal{S}pan_1\big) \cong \mathrm{ChEn}\big(\mathcal{S}pan(C), \mathcal{S}pan_1\big)^{\mathrm{op}} = \mathcal{S}pan_{ext}(C)^{\mathrm{op}}. \qquad \square$$

Remark 4.3.11. We give some interpretations regarding the significance of Theorem 4.3.10.

1. Intuitively, the natural transformation $\mathcal{S}pan \Rightarrow \mathcal{S}pan_{ext}$ exhibits $\mathcal{S}pan_{ext}$ as an "extension" of the bicategory $\mathcal{S}pan$.

2. The involution functor $\mathcal{S}pan_{ext}(C) \to \mathcal{S}pan_{ext}(C)^{\mathrm{op}}$ captures the obvious symmetry of $\mathcal{S}pan_{ext}(C)$ that is shared with $\mathcal{S}pan(C)$, since they only differ in their 2-cells.

3. For the same reason, the functor $\iota\colon C \hookrightarrow \mathcal{S}pan(C)$ from Remark 4.2.8 can be composed with the inclusion $\mathcal{S}pan(C) \hookrightarrow \mathcal{S}pan_{ext}(C)$ to obtain an inclusion functor $C \hookrightarrow \mathcal{S}pan_{ext}(C)$. It describes the obvious fact that C also "sits inside" the extended bicategory $\mathcal{S}pan_{ext}(C)$.
This can also be observed in a more abstract context. ι constitutes the components of a natural transformation from the inclusion functor $I\colon \mathcal{C}at_{pb} \hookrightarrow \mathcal{C}at_{pb}\text{-}\mathcal{C}at$ (viewing an ordinary category as a bicategory with trivial 2-cells) to $\mathcal{S}pan$. Composing this with the natural transformation $\mathcal{S}pan \Rightarrow \mathcal{S}pan_{ext}$ from Theorem 4.3.10 yields a natural transformation $I \Rightarrow \mathcal{S}pan_{ext}$ whose components are the inclusion functor $C \hookrightarrow \mathcal{S}pan_{ext}(C)$. ○

The above illustrates how various properties of the extended bicategory of spans $\mathcal{S}pan_{ext}(C)$ can be formally deduced from those of the bicategory of spans $\mathcal{S}pan(C)$, using the fact that the former arises from the latter by change of enrichment. This exemplifies how change of enrichment can shed light on the relationship between these two bicategories.

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